Research Report

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On an Infinite Series of Two Bessel Functions of First Kind

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THE stochastic and dynamical treatment of a system of coupled harmonic oscillators has achieved some successful results[1-10,13] on the bases of statistical theory of irreversible process. Up to the present, many investigators have mainly concerned with the case of an ***infinitely*** extended system, or at least with the case of a periodic system, so that mathematical difficulties can be avoided. In the ***finitely*** extended system, however, the dynamical recourence does not vanish and we should treat not only its dynamical recurrence but also its approach to the ***stationary state*** after a sufficiently long time.

Even within the scope of the one-dimensional system of harmonic oscillators—we call such a system as ***lattice-, we*** have still many problems to be solved. One of them is the convergence of infinite series of Bessel functions[4,6,7,9,10,13] and also the problem of uniform convergence[4,6,7,9,10,13] of some integrals involving Bessel functions. Here, in this note, we report the following formula which may be useful to treat some stochastic problems of one-dimensional lattice system[6,7,9,13] of ***finite*** length. We also notice at once that the stochastic and dynamical problems of the one-dimensional ***infinite*** lattice with some ***isotopic impurities***[4,7,9,13] can be conveniently treated by means of our formula of infinite series of Bessel functions of first kind.

It holds that

$$\sum_{r=-\infty}^{+\infty} p^{rN} J_{rN+\alpha}(z) J_{rN+\alpha+\beta}(z) =$$

$$= \frac{e^{-i\beta\pi/2}}{2Np^{(2\alpha+\beta)/2}} \cdot \sum_{k=0}^{2N-1} e^{-ik\pi(2\alpha+\beta)/N} \cdot J_\beta\{iz(p^{1/2} e^{ik\pi/N} - p^{-1/2} e^{-ik\pi/N})\} \qquad (1)$$

$$=\frac{e^{-i\beta\pi/2}}{2Np^{(2\alpha+\beta)/2}}\cdot\sum_{k=-N}^{N-1}e^{-ik\pi(2\alpha+\beta)/N}\cdot J_\beta\{iz(p^{1/2}e^{ik\pi/N}-p^{-1/2}e^{-ik\pi/N})\},\qquad(2)$$

where $J_\nu(z)$ is a Bessel function of first kind of order ν and argument z. N, a, and β are integers, and N>O. $i=\sqrt{-1}$.

In order to prove (1) and (2), we replace the product of Bessel functions in the summand of the left-hand side of (1) by the integral:

$$\frac{2}{\pi}\int_0^{\pi/2}d\theta\cdot J_{2rN+2\alpha+\beta}(2z\cos\theta)\cdot\cos\beta\theta,$$

and take into formula[13] nt the

$$\sum_{r=-\infty}^{+\infty}p^{rN}J_{rN+s}(z)=\frac{1}{Np^s}\cdot\sum_{k=0}^{N-1}\exp\left[\frac{z}{2}(pe^{2k\pi i/N}-p^{-1}e^{-2k\pi i/N})\right]\cdot e^{-2k\pi i/N}.\qquad(3)$$

In course of the calculation, we use the expression(") :

$$\exp\left[i\frac{\nu\pi}{2}\right]\cdot\int_0^{\pi/2}d\theta\cos\nu\theta\,.\exp\,[-iz\cos\theta]$$

$$=\frac{\pi}{2}[J_\nu(z)+iE_\nu(z)],\ \text{(for \&integer)}\qquad(4)$$

with $J_\nu(z)$ Anger's function of order ν and argument z, and $E_\nu(z)$ Weber's function of order ν and argument z. For any integer, n, Anger's function $J_n(z)$ is reduced- to Bessel function $J_n(z)$, while Weber's function $E_n(z)$ has a relation: $E_{-n}(z)=(-1)^nE_n(z)=-E_n(-z)$. Thus, the direct calculation easily leads to the formulae (1) and (2).

As a special case of (1), we put $p=1$ and $N=$(even integer) in (1), then the expression (1) is reduced to the result obtained in our previous paper.[11,12]

For $p=-b^2$ and $N=1$ in (1), we obtain:

$$\sum_{r=-\infty}^{+\infty}(-1)^rb^{2r}J_{r+\alpha}(z)J_{r+\alpha+\beta}(z)=\frac{(-1)^\alpha}{b^{2\alpha+\beta}}\cdot J_\beta\{z(b+\frac{1}{b})\},\qquad(5)$$

which is related to an addition formula:

$$J_\beta(x+y)=(-1)^\alpha\cdot\left(\frac{x}{y}\right)^{(2\alpha+\beta)/2}\cdot\sum_{r=-\infty}^{+\infty}(-1)^r\cdot\left(\frac{x}{y}\right)^r\cdot J_{r+\alpha}(\sqrt{xy})J_{r+\alpha+\beta}(\sqrt{xy}).\qquad(6)$$

In case of $\alpha=0$, the expression (5) is reduced to a formula derived- by Nielsen[15]:

$$\sum_{r=-\infty}^{+\infty}(-1)^rb^{2r}J_r(z)J_{r+\beta}(z)=\frac{1}{b^\beta}\cdot J_\beta\{z(b+\frac{1}{b})\}.\qquad(7)$$

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