

On the Bogoliubov-Zubarev-Taerkovnikov Method in the Theory of Superconductivity

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An alternative proof of the asymptotic exactness of the BCS theory of superconductivity within the framework of perturbation theory is made by employing the Bloch Statistical mechanical perturbation theory in the Bogoliubov-Zubarev-Tserkovnikov Method.

I. INTRODUCTION

THE Bardeen-Cooper-Schrieffer (BCS) model⁽¹⁾ of superconductivity is one of few examples in the many body problem that can be solved exactly. In the treatment of thermodynamic properties of superconductors the variational method was used in the original BCS theory⁽¹⁾. A different approach was developed by Bogoliubov, Zubarev and Tserkovnikov⁽²⁾, in which the problem was treated by the perturbation theory in statistical mechanics. It was shown by them that the solution of the BCS model is exact in the limit of infinite volume of superconducting system. In this paper we shall prove more rigorously the BZT method by using the statistical perturbation theory formulated by Bloch⁽³⁾. The virtue of using the Bloch theory is that the perturbation terms of each order of the grand partition function are calculable from a set of "Feynman" diagrams of the corresponding order. It must be added that the rigorous proof of the BZT method has been provided also by BZT⁽⁴⁾ by Green's function method avoiding the controversial perturbation method. However it is the opinion of the author that the exact solution of the BCS model by the original BZT perturbation method may set a light to the origin of the infinity difficulty encountered in the conventional field theory.

II. BOGOLIUBOV'S CANONICAL TRANSFORMATION

Following BCS the Hamiltonian of an electron system in superconductor can be written in the following form:

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- (1) J. Bardeen, L. N. Cooper and J. R. Schrieffer, Phys. Rev. 106, 162 (1957); *ibid* 108, 1175 (1957).
 - (2) N. N. Bogoliubov, D. N. Zubarev and Yu. A. Tserkovnikov, Soviet Physics Doklady 2, 535 (1958).
 - (3) C. Bloch, Nuclear Physics, 7, 451 (1958).
 - (4) N. N. Bogoliubov, D. N. Zubarev and Yu. A. Tserkovnikov, Soviet Physics JETP 12, 88 (1961).

$$H = \sum_{\mathbf{k}} e(\mathbf{k}) (C_{\mathbf{k}\uparrow}^+ C_{\mathbf{k}\uparrow} + C_{\mathbf{k}\downarrow}^+ C_{\mathbf{k}\downarrow}) - \sum_{\mathbf{k}} \sum_{\substack{\mathbf{k}' \\ (\mathbf{k} \neq \mathbf{k}')}} \frac{J(\mathbf{k}', \mathbf{k})}{V} C_{\mathbf{k}'\uparrow}^+ C_{-\mathbf{k}'\downarrow}^+ C_{\mathbf{k}\uparrow} C_{-\mathbf{k}\downarrow}. \quad (1)$$

Here $e(\mathbf{k})$ is the Bloch electron corresponding to the wave vector \mathbf{k} . $J(\mathbf{k}', \mathbf{k})$ is a small positive quantity symmetric in \mathbf{k} and \mathbf{k}' and nonvanishing only for those \mathbf{k}' and \mathbf{k} whose corresponding energies $e(\mathbf{k}')$ and $e(\mathbf{k})$ lie in the neighborhood of the Fermi energy μ of the system. V is the volume of the system. The creation and annihilation operators, $C_{\mathbf{k}\sigma}^+$ and $C_{\mathbf{k}\sigma}$ (the index σ denotes either spin-up, \uparrow , or spin-down, \downarrow), satisfy the, following anti-commutation relations:

$$\{C_{\mathbf{k}\sigma}, C_{\mathbf{k}'\sigma'}^+\} = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}, \{C_{\mathbf{k}\sigma}, C_{\mathbf{k}'\sigma'}\} = 0. \quad (2)$$

For statistical mechanical consideration the following quantity is of interest:

$$\tilde{H} = H - \mu N, \quad (3)$$

where N is the particle number operator given by

$$N = \sum_{\mathbf{k}} (C_{\mathbf{k}\uparrow}^+ C_{\mathbf{k}\uparrow} + C_{\mathbf{k}\downarrow}^+ C_{\mathbf{k}\downarrow}). \quad (4)$$

Our whole problem will be to diagonalize \tilde{H} to the extent that the residual perturbation terms do not contribute to thermodynamic functions of the system in the limit of large volume V . To accomplish this purpose the following Bogoliubov canonical transformation can be made:

$$\begin{aligned} b_{\mathbf{k}\uparrow} &= u_{\mathbf{k}} C_{\mathbf{k}\uparrow} - v_{\mathbf{k}} C_{-\mathbf{k}\downarrow}^+, \\ b_{\mathbf{k}\downarrow} &= u_{\mathbf{k}} C_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} C_{\mathbf{k}\uparrow}^+, \end{aligned} \quad (5)$$

where $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$ and the new operators $b_{\mathbf{k}\sigma}$, $b_{\mathbf{k}\sigma}^+$, satisfy the same anticommutation relations as $C_{\mathbf{k}\sigma}$, $C_{\mathbf{k}\sigma}^+$ in (2).

By (5) the expression (3) becomes

$$\tilde{H} = \tilde{H}_0 + \tilde{H}', \quad (6)$$

where

$$\begin{aligned} \tilde{H}_0 &= \sum_{\mathbf{k}} [2 \varepsilon(\mathbf{k}) v_{\mathbf{k}}^2 - u_{\mathbf{k}} v_{\mathbf{k}} \varepsilon_0(\mathbf{k})] + X(\mathbf{k}) (b_{\mathbf{k}\uparrow}^+ b_{\mathbf{k}\uparrow} + b_{\mathbf{k}\downarrow}^+ b_{\mathbf{k}\downarrow}) \\ &\quad + Y(\mathbf{k}) (b_{\mathbf{k}\uparrow}^+ b_{\mathbf{k}\downarrow}^+ + b_{\mathbf{k}\downarrow} b_{\mathbf{k}\uparrow}), \end{aligned} \quad (7)$$

$$\varepsilon(\mathbf{k}) \equiv e(\mathbf{k}) - \mu,$$

$$\varepsilon_0(\mathbf{k}) = \sum_{\mathbf{k}'} \frac{J(\mathbf{k}, \mathbf{k}')}{V} u_{\mathbf{k}'} v_{\mathbf{k}'},$$

$$X(\mathbf{k}) = \varepsilon(\mathbf{k}) (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) + 2u_{\mathbf{k}} v_{\mathbf{k}} \varepsilon_0(\mathbf{k}),$$

$$Y(\mathbf{k}) = \varepsilon_0(\mathbf{k}) (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) - 2u_{\mathbf{k}} v_{\mathbf{k}} \varepsilon(\mathbf{k}),$$

$$\tilde{H}' = - \sum_{\mathbf{k}} \sum_{\substack{\mathbf{k}' \\ (\mathbf{k} \neq \mathbf{k}')}} \frac{J(\mathbf{k}', \mathbf{k})}{V} B_{\mathbf{k}'}^+ B_{\mathbf{k}}, \quad (8)$$

$$B_k = u_k v_k (b_{k\uparrow}^\dagger b_{k\uparrow} + b_{k\downarrow}^\dagger b_{k\downarrow}) - u_k^2 b_{k\downarrow} b_{k\uparrow} + v_k^2 b_{k\uparrow}^\dagger b_{k\downarrow}^\dagger.$$

The parameters u_k or v_k introduced in (5) will be determined in the next section so as to make the contribution of the perturbation \tilde{H}' infinitely small compared with \tilde{H}_0 in the limit of large volume.

Our next task is to diagonalize \tilde{H}_0 in (7). Again we perform the following canonical transformation :

$$\begin{aligned} a_{k\uparrow} &= \lambda_k b_{k\uparrow} - \nu_k b_{k\downarrow}^\dagger, \\ a_{k\downarrow} &= \lambda_k b_{k\downarrow} + \nu_k b_{k\uparrow}^\dagger, \end{aligned} \quad (9)$$

where $a_{k\sigma}$ satisfy the anticommutation relations of (2) and $\lambda_k^2 + \nu_k^2 = 1$. By a straightforward calculation it is found that \tilde{H}_0 is diagonalized if the parameters λ_k and ν_k are taken to be

$$\begin{aligned} \lambda_k &= \frac{1}{\sqrt{2}} \left[1 \pm \frac{X(k)}{\sqrt{X^2(k) + Y^2(k)}} \right]^{\frac{1}{2}}, \\ \nu_k &= \frac{1}{\sqrt{2}} \left[1 \mp \frac{X(k)}{\sqrt{X^2(k) + Y^2(k)}} \right]^{\frac{1}{2}}, \end{aligned} \quad (10)$$

where the upper (lower) sign is taken for those $\varepsilon(k) = e(k) - \mu$ larger (smaller) than zero.

The \tilde{H} is then given by

$$\tilde{H} = \tilde{H}_0 + \tilde{H}', \quad (11)$$

$$H_0 = U_0 + \sum_k E(k) (a_{k\uparrow}^\dagger a_{k\uparrow} + a_{k\downarrow}^\dagger a_{k\downarrow}), \quad (12)$$

$$U_0 = \sum_k [\varepsilon(k) - E(k) + u_k v_k \varepsilon_0(k)], \quad (13)$$

$$E(k) = \pm \sqrt{\varepsilon^2(k) + \varepsilon_0^2(k)}, \quad (14)$$

$$\tilde{H}' = - \sum_{\substack{k, k' \\ (k' \neq k)}} \frac{J(k', k)}{V} B_{k'}^\dagger B_k, \quad (15)$$

$$\begin{aligned} B_k &= \left(u_k v_k - \frac{\varepsilon_0(k)}{2E(k)} \right) + \frac{1}{2} \left(1 - \frac{\varepsilon(k)}{E(k)} \right) a_{k\uparrow}^\dagger a_{k\downarrow}^\dagger \\ &\quad + \frac{1}{2} \left(1 + \frac{\varepsilon(k)}{E(k)} \right) a_{k\uparrow} a_{k\downarrow} + \frac{1}{2} \frac{\varepsilon_0(k)}{E(k)} (a_{k\uparrow}^\dagger + a_{k\downarrow}^\dagger). \end{aligned} \quad (16)$$

Thus the electron system can be regarded as a system of quasi-particles with energy spectrum $\mathbf{E}(\mathbf{k})$ in (14) characterized by an energy gap $2\varepsilon_0(k)$ near the Fermi surface.

III. BLOCH'S STATISTICAL PERTURBATION THEORY

In the previous section we were able to write \tilde{H} into two parts: the unperturbed part, \tilde{H}_0 , which is diagonalized, and the perturbation \tilde{H}' . Our problem now

is to calculate the grand partition function Z of the electron system defined by

$$Z = \text{Tr} [e^{-\beta \tilde{H}}]. \quad (17)$$

Here $\beta = 1/kT$, T being temperature and k being Boltzmann's constant.

Tr denotes the taking of the trace of the exponential.

If the perturbation \tilde{H}' can be neglected the grand partition function Z_0 of the system is easily solved, namely

$$Z_0 = \text{Tr} [e^{-\beta \tilde{H}_0}] = e^{-\beta U_0} \prod_k (1 + e^{-\beta E(k)})^2. \quad (18)$$

U_0 and $E(k)$ are given in (13) and (14).

If the effect of the perturbation \tilde{H}' is taken into consideration we would like to expand Z_0 into a series in powers of $J(k, k')$ in \tilde{H}' . In the following we adapt the Bloch's formalism⁽³⁾.

Let us define an operator $S(\beta)$ by

$$e^{-\beta \tilde{H}} = e^{-\beta \tilde{H}_0} S(\beta). \quad (19)$$

The operator $S(\beta)$ can be shown to satisfy the following differential equation and initial condition:

$$\frac{\partial S(\beta)}{\partial \beta} = -\tilde{H}'(\beta) S(\beta), \quad S(0) = 1, \quad (20)$$

where $\tilde{H}'(\beta)$ is defined by

$$\tilde{H}'(\beta) = e^{\beta \tilde{H}_0} \tilde{H}' e^{-\beta \tilde{H}_0}. \quad (21)$$

By iteration procedure, (20) can be solved and the following power series expansion of $S(\beta)$ is obtained:

$$S(\beta) = \sum_{n=0}^{\infty} (-)^n \int_0^\beta dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \tilde{H}'(t_1) \tilde{H}'(t_2) \dots \tilde{H}'(t_n). \quad (22)$$

Let $\langle A \rangle$ be the statistical average of the operator A , defined by

$$\langle A \rangle \equiv \text{Tr} [A e^{-\beta \tilde{H}_0}] / \text{Tr} [e^{-\beta \tilde{H}_0}]. \quad (23)$$

Then from (17), (18) and (19), we get

$$Z = Z_0 \langle S(\beta) \rangle. \quad (24)$$

Let \mathcal{Q} and \mathcal{Q}_0 be the Gibbs thermodynamic functions of the unperturbed and perturbed systems respectively. From the definition

$$Z = e^{-\beta \mathcal{Q}} \quad \text{and} \quad Z_0 = e^{-\beta \mathcal{Q}_0},$$

we get from (24) the following expression for \mathcal{Q} :

$$\mathcal{Q} = \mathcal{Q}_0 - \frac{1}{\beta} \ln \langle S(\beta) \rangle. \quad (25)$$

The Ω_0 is immediately obtained from (18), *i.e.*

$$\Omega_0 = -\frac{1}{\beta} \ln Z_0 = \sum_{\mathbf{k}} \left[\varepsilon(\mathbf{k}) - E(\mathbf{k}) + u_{\mathbf{k}} v_{\mathbf{k}} \varepsilon_0(\mathbf{k}) - \frac{2}{\beta} \ln(1 + e^{-\beta E(\mathbf{k})}) \right] \quad (26)$$

The solution of the next term $-\frac{1}{\beta} \ln \langle S(\mathbf{B}) \rangle$ in (25) is the main task of the present section. The reader is reminded that this term will be shown to yield null contribution in the limit of large volume of the system if proper values of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are chosen, so that the solution

$$\Omega = \Omega_0 \quad (27)$$

is asymptotically exact. In fact (26) gives all the thermodynamic properties of superconductors.

Now we start to compute $\langle S(\mathbf{P}) \rangle$,

$$\langle S(\beta) \rangle = \sum_{n=0}^{\infty} (-)^n \int_0^{\beta} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \langle \tilde{H}'(t_1) \tilde{H}'(t_2) \dots \tilde{H}'(t_n) \rangle, \quad (28)$$

by Feynman's diagram technique used in the field theory.⁽⁵⁾

Consider first (from (15) and (21))

$$\tilde{H}'(t) = - \sum_{(\mathbf{k} \approx \mathbf{k}')} \sum \frac{J(\mathbf{k}', \mathbf{k})}{V} \tilde{B}_{\mathbf{k}'}(t) B_{\mathbf{k}}(t), \quad (29)$$

where

$$\begin{aligned} B_{\mathbf{k}}(t) &= e^{t \tilde{H}_0} B_{\mathbf{k}} e^{-t \tilde{H}_0}, \\ \tilde{B}_{\mathbf{k}}(t) &= e^{t \tilde{H}_0} B_{\mathbf{k}}^{\pm} e^{-t \tilde{H}_0}. \end{aligned} \quad (30)$$

Since it can be easily shown that

$$\begin{aligned} e^{t H_0} a_{\mathbf{k}\sigma}^{\pm} e^{-t \tilde{H}_0} &= a_{\mathbf{k}\sigma}^{\pm} e^{t E(\mathbf{k})}, \\ e^{t \tilde{H}_0} a_{\mathbf{k}\sigma} e^{-t \tilde{H}_0} &= a_{\mathbf{k}\sigma} e^{-t E(\mathbf{k})}, \quad \sigma = \uparrow \text{ or } \downarrow, \end{aligned}$$

we get from (30) and (16)

$$\begin{aligned} B_{\mathbf{k}}(t) &= \left(u_{\mathbf{k}} v_{\mathbf{k}} - \frac{\varepsilon_0(\mathbf{k})}{2E(\mathbf{k})} \right) + \frac{1}{2} \left(1 - \frac{\varepsilon(\mathbf{k})}{E(\mathbf{k})} \right) a_{\mathbf{k}\uparrow}^{\dagger} a_{\mathbf{k}\downarrow}^{\dagger} e^{2t E(\mathbf{k})} \\ &\quad + \frac{1}{2} \left(1 + \frac{\varepsilon(\mathbf{k})}{E(\mathbf{k})} \right) a_{\mathbf{k}\uparrow} a_{\mathbf{k}\downarrow} e^{-2t E(\mathbf{k})} + \frac{\varepsilon_0(\mathbf{k})}{2E(\mathbf{k})} (a_{\mathbf{k}\uparrow}^{\dagger} a_{\mathbf{k}\uparrow} + a_{\mathbf{k}\downarrow}^{\dagger} a_{\mathbf{k}\downarrow}), \\ \tilde{B}_{\mathbf{k}}(t) &= \left(u_{\mathbf{k}} v_{\mathbf{k}} - \frac{\varepsilon_0(\mathbf{k})}{2E(\mathbf{k})} \right) - \frac{1}{2} \left(1 - \frac{\varepsilon(\mathbf{k})}{E(\mathbf{k})} \right) a_{\mathbf{k}\uparrow} a_{\mathbf{k}\downarrow} e^{-2t E(\mathbf{k})} \\ &\quad - \frac{1}{2} \left(1 + \frac{\varepsilon(\mathbf{k})}{E(\mathbf{k})} \right) a_{\mathbf{k}\uparrow}^{\dagger} a_{\mathbf{k}\downarrow}^{\dagger} + \frac{1}{2} \frac{\varepsilon_0(\mathbf{k})}{E(\mathbf{k})} (a_{\mathbf{k}\uparrow}^{\dagger} a_{\mathbf{k}\uparrow} + a_{\mathbf{k}\downarrow}^{\dagger} a_{\mathbf{k}\downarrow}). \end{aligned} \quad (31)$$

Substituting (31) into (29) we obtain that $\tilde{H}'(t)$ contains ten different combinations of $a_{\mathbf{k}\sigma}$ operators for each set of \mathbf{k} and \mathbf{k}' . Besides the common factor $-\frac{J(\mathbf{k}', \mathbf{k})}{V}$,

(5) N.H. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Field*, (Interscience Publishers, Inc., New York, 1959.)

these ten combinations of $a_{k\sigma}$ operators and their coefficients are shown in Table 1. Each operator term can be represented by a suitable Feynman diagram. The common factor $\frac{-J(k', k)}{V}$ is represented by a dot “•”. Each annihilation operator $a_{k\sigma}$ is represented by a line with an arrow sign pointing inward to the dot, while for each creation operator $a_{k\sigma}^+$ a line with an arrow pointing outward from the dot is drawn. For the diagonal term, $a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}$, a closed circle connecting the dot is represented. The diagrams of the ten operator terms of $\tilde{H}'(t)$ are indicated in the last column of the Table 1.

Table 1. The operator terms in $\tilde{H}'(t)$ for fixed k and k' and their coefficients besides the common factor $-J(k, k')/V$. The corresponding Feynman diagrams are shown in the last column.

| # | Operate Term | Coefficient | Diagram |
|----|---|---|---------|
| 1 | 1 | $(u_k v_k - \frac{\epsilon_0(k)}{2E(k)}) (u_{k'} v_{k'} - \frac{\epsilon_0(k')}{2E(k')})$ | |
| 2 | $a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}$ | $\frac{\epsilon_0(k)}{E(k)} (u_k v_{k'} - \frac{\epsilon_0(k')}{2E(k')})$ | |
| 3 | $a_{k\uparrow} a_{k\downarrow}$ | $\frac{\epsilon(k)}{E(k)} (u_k v_{k'} - \frac{\epsilon_0(k')}{2E(k')}) e^{-2tE(k)}$ | |
| 4 | $a_{k\uparrow}^+ a_{k\downarrow}^+$ | $-\frac{\epsilon(k)}{E(k)} (u_k v_{k'} - \frac{\epsilon_0(k')}{2E(k')}) e^{2tE(k)}$ | |
| 5 | $(a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}) \cdot (a_{k'\uparrow}^+ a_{k'\uparrow} + a_{k'\downarrow}^+ a_{k'\downarrow})$ | $\frac{\epsilon_0(k) \epsilon(k')}{4E(k) E(k')}$ | |
| 6 | $(a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}) a_{k'\uparrow}^+ a_{k'\downarrow}^+$ | $-\frac{\epsilon_0(k) \epsilon(k')}{2E(k) E(k')} e^{+2tE(k)}$ | |
| 7 | $(a_{k\uparrow}^+ a_{k\uparrow} + a_{k\downarrow}^+ a_{k\downarrow}) a_{k'\uparrow} a_{k'\downarrow}$ | $\frac{\epsilon_0(k) \epsilon(k')}{2E(k) E(k')} e^{-2tE(k)}$ | |
| 8 | $a_{k\uparrow} a_{k\downarrow} a_{k'\uparrow}^+ a_{k'\downarrow}^+$ | $\frac{1}{2} \left(1 + \frac{\epsilon(k) \epsilon(k')}{E(k) E(k')} \right) e^{2t(E(k') - E(k))}$ | |
| 9 | $a_{k\uparrow} a_{k\downarrow} a_{k'\downarrow} a_{k'\uparrow}$ | $-\frac{1}{4} \left(1 - \frac{\epsilon(k) \epsilon(k')}{E(k) E(k')} \right) e^{-2t(E(k) + E(k'))}$ | |
| 10 | $a_{k\uparrow}^+ a_{k\downarrow}^+ a_{k'\uparrow} a_{k'\downarrow}$ | $-\frac{1}{4} \left(1 - \frac{\epsilon(k) \epsilon(k')}{E(k) E(k')} \right) e^{2t(E(k) + E(k'))}$ | |

Now we are in the position of evaluating $\langle S(\beta) \rangle$ in (28). The contribution from the n -th order $\langle S_n(\beta) \rangle$,

$$\langle S_n(\beta) \rangle = (-)^n \int_0^\beta dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n \langle \tilde{H}'(t_1) \tilde{H}'(t_2) \dots \tilde{H}'(t_n) \rangle, \quad (32)$$

can be computed from a set of Feynman diagrams possibly constructed from any n diagrams in Table 1 by appropriately connecting arrow-directed lines of different vertex dots, ordered from the bottom to top in increasing value of the "time" t . For example the two diagrams in Fig. 1.

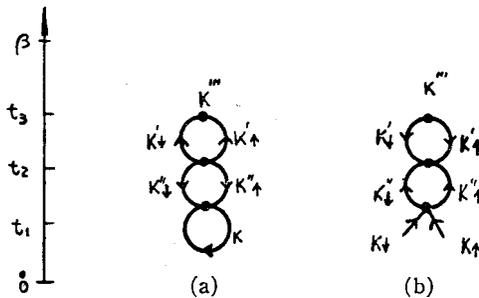


Fig. 1 Two of third order diagrams

are two of the third order diagrams contributing to $\langle \tilde{H}'(t_1) \tilde{H}'(t_2) \tilde{H}'(t_3) \rangle$. The first one is the combination of the diagrams # 7, # 10 and # 3. The second one is the combination of the diagrams #8, #9 and #4. The arrowed lines connecting two different vertices (dots) are said to be "contracted". The contracted lines in Fig. 1 are those labelled by k' and k'' .

The question now is how to write down the contribution of each diagram explicitly in analytic expression. The general discussion of this problem was made by Bloch and Dominicis⁽⁶⁾ and is elaborated in standard textbook in field theory⁽⁵⁾. To make a long story short we here only present the final results of discussion and elaboration. First of all we notice that diagrams (like Fig. 1 (b)) with "external lines", i. e. uncontracted lines do not contribute. Hence only "vacuum-vacuum" diagrams such as Fig: 1 (a) contribute. The rules of explicitly writing down the analytic contribution of a "vacuum-vacuum" diagram are as follows:

1). Each vertex (dot) contributes a factor $\frac{J(k', k)}{V}$ and a coefficient in Table 1 depending on the kind of vertex which is shown in the Table 1.

2). A contracted line between two different vertices (at different "time" t) contributes a factor n_k given by

$$n_k = \frac{1}{e^{\beta E(k)} + 1}, \quad (33)$$

(6) C. Bloch and D. De Dominicis, Nuclear Physics **7**, 459 (1958).

if its arrow points downward (to the direction of negative "time"), and a factor $1 - n_k$, if its arrow points upward (to the direction of positive "time").

3). A contracted line to the same vertex (at the same "time") contributes a factor $2n_k$.

4). Multiplying all factors in 1), 2) and 3) together, and summing over all k and k' at each vertex.

5). Integrating over the "time" t of each vertex such that each ordered t_i in $\beta > t_1 > t_2 > \dots > t_i > t_{i+1} \dots > t_n > 0$, is integrated from 0 to t_{i-1} .

A simple example is the contribution of the diagram in Fig. 1 (a), which is as follow:

$$\int_0^\beta dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \sum_{\substack{k''', k' \\ (k''' \neq k')}} \sum_{\substack{k' \\ (k' \neq k''')}} \sum_{\substack{k'' \\ (k'' \neq k)}} \sum_{\substack{k \\ (k \neq k')}} \frac{J(k''', k')}{V} \frac{J(k', k'')}{V} \frac{J(k'', k)}{V} \\ (1 - n_{k'}) (1 - n_{k''}) \cdot n_{k'''} n_{k''} \cdot 2n_k \left\{ \frac{\varepsilon_0(k')}{E(k')} \left(u_{k'''} v_{k''} - \frac{\varepsilon_0(k''')}{2E(k''')} \right) e^{-2t_3 E(k')} \right\} \cdot \\ \cdot \left\{ -\frac{1}{4} \left(1 - \frac{\varepsilon(k')}{E(k')} \cdot \frac{\varepsilon(k'')}{E(k'')} \right) e^{2t_2 (E(k') + E(k''))} \right\} \left\{ \frac{\varepsilon_0(k) \varepsilon(k'')}{2E(k) E(k'')} e^{-2t_1 E(k'')} \right\}.$$

In summary the contribution to $\langle S(P) \rangle$ of each order perturbation can be calculated by first drawing all possible Feynman diagrams of that order and then writing down their expressions by the above rules. However Bloch⁽³⁾ has shown that $\langle S(\beta) \rangle$ can also be written as

$$\langle S(\beta) \rangle = \exp [\langle S(\beta) \rangle_c], \tag{34}$$

where c stands for taking only those vacuum-vacuum diagrams that are "connected". A diagram is said to be "disconnected" if part of the diagram has no lines connecting with the other parts of the diagram. For example the diagram in Fig. 2 is a fourth order disconnected diagram.

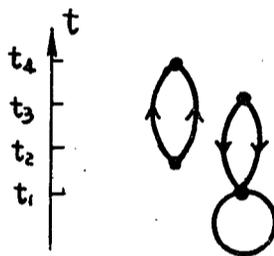


Fig. 2 One of the 4th order disconnected diagrams

Hence we need only to consider the connected vacuum-vacuum diagrams. The thermodynamic function Ω in (25) is thus simplified to

$$\Omega = \Omega_0 - \frac{1}{\beta} \langle S(\beta) \rangle_c. \tag{35}$$

The n -th order correction of \mathcal{Q} is therefore contributed by the n -th order connected vacuum diagrams $\langle S_n(\beta) \rangle_c$.

IV. EXACT SOLUTION OF THE BCS MODEL

In this section we shall show that, by suitable choice of the parameters u_k and v_k introduced in (5), \mathcal{Q}_0 in (35) is the asymptotically exact expression for the thermodynamic function \mathcal{Q} , *i. e.* the perturbation terms in $\langle S(\beta) \rangle_c$ tend to be infinitesimally small compared with \mathcal{Q}_0 as the volume $V \rightarrow \infty$.

First of all we consider the first perturbation term in $\langle \mathbf{S}(\mathbf{B}) \rangle_c$, *i. e.* $\langle S_1(\beta) \rangle_c$ which according to the rules in the previous section is given by

$$\begin{aligned} \langle S_1(\beta) \rangle_c &\equiv - \int_0^\beta dt_1 \langle \tilde{H}'(t_1) \rangle_c = \left(\begin{array}{c} k' \\ \circ \\ k \end{array} + \begin{array}{c} k \\ \circ \\ k \end{array} + \begin{array}{c} k \\ \circ \\ k \end{array} \right) \\ &= \beta \sum_k \sum_{\substack{k' \\ (k \neq k')}} \frac{J(k, k')}{V} \left[u_k v_k - \frac{\varepsilon_0(k)}{2E(k)} \operatorname{tgh} \frac{1}{2} \beta E(k) \right] \left[u_{k'} v_{k'} - \frac{\varepsilon_0(k')}{2E(k')} \operatorname{tgh} \frac{1}{2} \beta E(k') \right] \end{aligned}$$

In order that the perturbation in (35) be small we set

$$\langle S_1(\beta) \rangle_c = 0$$

which gives the following condition for u_k and v_k :

$$u_k v_k = \frac{\varepsilon_0(k)}{2E(k)} \operatorname{tgh} \frac{1}{2} \beta E(k). \quad (36)$$

The condition (36) together with $u_k^2 + v_k^2 = 1$ determines u_k and v_k . From (7) we obtain the famous gap equation:

$$\varepsilon_0(k) = \sum_{k'} \frac{J(k, k')}{V} \cdot \frac{\varepsilon_0(k')}{2E(k')} \operatorname{tgh} \frac{1}{2} \beta E(k'). \quad (37)$$

Also from (36) we get

$$\mathcal{Q}_0 = \sum_k \left[\varepsilon(k) - E(k) + \frac{\varepsilon_0^2(k)}{2E(k)} \operatorname{tgh} \frac{1}{2} \beta E(k) - \frac{2}{\beta} \ln(1 + e^{-\beta E(k)}) \right] \quad (38)$$

The expressions (37) and (38) are basic equations in the theory of superconductivity, from which all important thermodynamic properties of superconductors are derived.

Now the important consequence of the condition (36) is that, not only it gives rise to the vanishment of the first order perturbation term of \mathcal{Q} , but also makes the higher order perturbation terms small compared with the unperturbed term \mathcal{Q}_0 . To say more precisely, all the higher order perturbation terms of \mathcal{Q} as the consequence of the condition (36) are at best independent of the volume V of the system, while the leading term \mathcal{Q}_0 is proportional to V . The detail proof of this statement will be given in the followings.

As a simple illustration of the general proof to be given later, we first consider the second order perturbation terms, $\langle S_2(\beta) \rangle_c$. All the second order diagrams are exhausted in the Fig. 3.

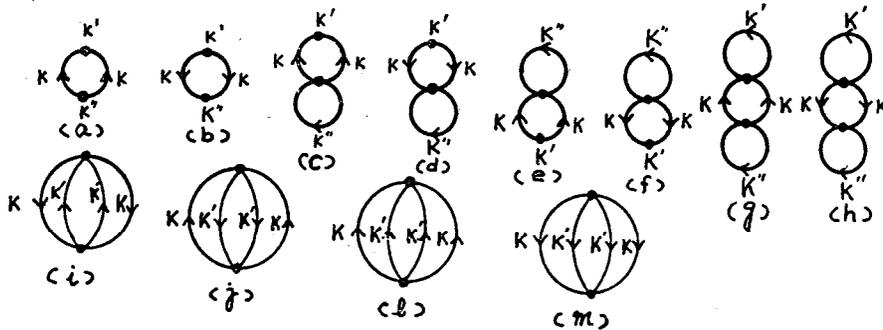


Fig. 3 The second order Feynman diagrams

There are two points that must be remarked. Firstly the first eight diagrams in Fig.3(from (a) to (h)) are the combinations of two of the basic diagrams # 3, #4, #6, and #7 of Table 1. There are three k indices (k, k', k'') that are to be summed, which gives rise to a factor V^3 . Since with each vertex is associated a factor $J(k, k')/V$, there is a factor $1/V^2$ contributed from the two vertices. Hence the contributions of these first eight diagrams are proportional to V_0 , which is comparable to Ω_0 . Secondly the last four diagrams in Fig. 3 (from (i) to (m)) are combinations of two of the basic diagrams #8, #9 and #10 of Table 1, There are only two k indices that are to be summed, hence the summations give rise to a factor V^2 which is cancelled by the factor $1/V^2$ associated with the two vertices. Hence the contributions of the last four diagrams of Fig. 3 are independent of V , which therefore is of the order of $1/V$ compared with Ω_0 .

Now consider the four diagrams (a), (c), (e) and (g) of Fig. 3. These four diagrams have a common feature, that is, they may be considered as constructed from a "body"

$$\text{Diagram (a)} \quad (39)$$

by adding to its "open vertices" either dots, ".", or "loops",

$$\text{Diagram (c)}$$

or both. That is, if we add two dots to the "body" we get diagram (a) ; if one dot and one loop, we get diagrams (c) and (e) ; if two loops we get diagram (g).

Now the "body" contributes the same factor to the four diagrams under consideration, i. e, it contributes a factor

is zero and therefore the contribution of the second order diagrams to \mathcal{Q} is at best of the order of $1/V$ with respect to \mathcal{Q}_0 .

The above discussions on the second order diagrams can easily be extended to diagrams of higher orders. A diagram of any order that consists of at least two basic diagrams of #3, #4, #6 and #7 of Table 1 can be schematically shown by one of the four diagrams of Fig. 4. (Note that these basic diagrams occur only in pairs)

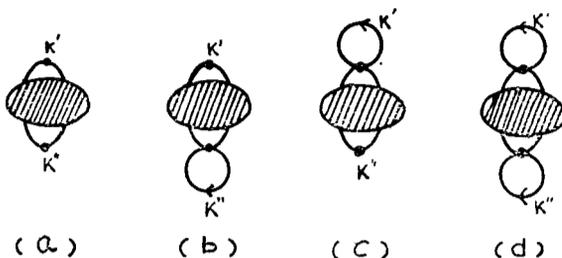


Fig. 4 Schematical structure of diagrams consisting of two or more basic diagrams of #3, #4, #6 and #7 of Table 1. denotes the rest of the diagrams. The vertices k' and k'' need not be at the highest lowest "time".

It is always possible to pick up among diagrams of any order four diagrams of Fig. 4 with a common "body" structure (with the two vertices "open") :

(43)

The contributions of these four diagrams, besides the same factor from the common "body" diagram (43), are obviously those of (41). Hence according to (42) the sum of these four diagrams gives zero contribution.

We thus conclude that for diagrams of any order those which consist of basic diagrams of #3, #4, #6 and #7 of Table 1 give zero contribution, therefore only those which are constructed out of the three basic diagrams of #8, #9 and #10 of Table 1 contribute. We now show in the followings that these diagrams contribute quantities of order 1 or less with respect to the volume V .

We consider first the second order diagrams $(i), (j), (l)$ and (m) of Fig. 3. Each diagram consists of two vertices, each having two pairs of outgoing or incoming lines. Each pair of outgoing or incoming lines carries with it a K -variable. Hence originally there are 4 K -variables. But the contraction of lines cuts the number into half, therefore there are only two K -variables in these diagrams, which contribute a factor V^2 . This factor is cancelled by another factor $1/V^2$ associated with two $J(k, k')$ of the two vertices, hence the contribution of these diagrams is of the order of 1 or less with respect to V .

Now we consider in general an n -th order diagram built up by the basic diagrams of #8, #9 and #10 of Table 1. There are n vertices each of which has two pairs of outgoing or incoming lines carrying two k -variables, k_i and k_i' , $i=1, 2, \dots, n$. The contraction of these $2n$ pairs of lines diminishes the number of k -variables to half of the original number $2n$ or less. Thus if to each $k_i(k_i')$ there is exactly one $k_j(k_j')$, $i \neq j$, such that $k_i(k_i') = k_j(k_j')$ *i. e.* $k_i(k_i')$ and $k_j(k_j')$ are contracted, then $2n/2 = n$ k -variables exist. The summation over these n k -variables in the contribution of the diagram gives rise to a factor V^n which will be cancelled by n factors of $1/V$ constantly attached to $J(\mathbf{k}, \mathbf{k}')$ of each vertex. Hence the contribution of the diagram is of the order $V^n \cdot (1/V)^n = 1$. If to each $k_i(k_i')$ more than one $k_j(k_j')$, $j \neq i$, which are contracted to $k_i(k_i')$, *i. e.* $k_j(k_j') = k_i(k_i')$, then the number m of the k -variables becomes less than n , *i. e.* $m < n$. Hence the contribution of diagrams of this kind is of the order of $V^m(1/V)^n = V^{m-n} \leq 1/V$, at least one order of magnitude smaller than that of the first case. Fig. 5 gives examples of the nonvanishing 4-th order diagrams. (a) contributes a value of order 1, while (b) contributes a value of order $1/V$.

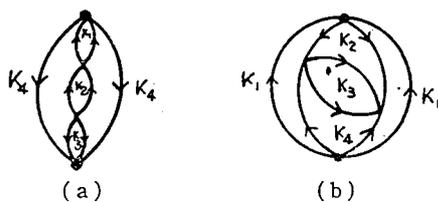


Fig. 5 Two of the 4-th order diagrams.

- (a) Contributes a value of order 1.
- (b) Contributes a value of order $1/V$.

In conclusion we have shown that the perturbation terms of each order in $\langle S(\beta) \rangle_c$ contribute to Ω values only of the order of 1 or less with respect to V . Since the perturbation series of $\langle S(\beta) \rangle_c$ is presumably convergent, the contribution of $\langle S(\beta) \rangle_c$ in (35) to Ω is therefore of the order of 1 which is infinitesimally small compared with Ω_0 (which is proportional to V) in the limit of large V . Thus we have proved that Ω_0 given in (38) is asymptotically exact expression for the thermodynamic function of superconducting system.