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Stochastic Transients of Noisy Lotka-Volterra Model

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A Lotka-Volterra model system perturbed by external noises is investigated in detail. Spatial homogeneity is assumed over the reacting system. External white noises are realized either as additional random forces in rate equations, or as rapid fluctuating parts of various rate parameters. Transient properties are described in terms of the average concentrations together with the fluctuations and correlation among them. The system which is marginally stable in deterministic theory, turns out to be unstable due to noises. Fluctuations are found to be increasing with noise intensities. For a given value of noise intensity, fluctuations are oscillating and incresing monotonically with time. The closed phase trajectories are found to be spiraling toward absorbing boundaries. Approximate period can be determined numerically to demonstrate the noise-induced slowing-down phenomenon. It is found that period always increases with noise intensity, and that successive cycles take longer and longer period. Due to fluctuations, a Lyapunov function of the system is found to be more positive and oscillating with time. The time rate of change of this function is derived analytically, and is found to have distinct contributions from the intrinsic and extrinsic stochasticities.

I. INTRODUCTION

Complex systems in Physics, Chemistry and Biology share similarity as well as diversity.[1,2] All these systems consist of enomous number of interacting constituents, and are usually described by a set of deterministic equations involving a small number of variables.[3,4] Rate constants appear in the deterministic equations are in fact average quantities which represent a series of complex processes, and therefore contain characteristics of the dynamic systems as well as environmental constraints. Investigations of intrinsic stochasticity and extrernal noise effects should be considered as an essential part in understanding the complex systems.[s-7]

Studies of noise effect have two main purposes. The one is to estimate the extent to which a given noise might influence the system. The other one is to probe the possible noise-induced phenomena which are of physical interest.[8] Investigation of the noise-induced transition phenomena in nonequilibrium systems will compliment to the theory of phase transition in general.

Noises usually cause destructive effects in such a way that the system ' s stability is weaken,[8] and that fluctuation catastrophe might arise since noises use to increase fluctuation and slow down the transient process.' However it had also been found that some noises might stabilize systems,[10] sustain oscillations[11] or induce stable steady states.[12] It is interesting to note that most of neural networks, which are also complex interacting systems, are very much noise-robust, and that noise plays crucial role in the operation of the neural systems.[13] Much more efforts are needed in order to have a better understanding of the nature of noise effects on complex systems.

Even for one-variable system, stochastic study is a formidable task due to nonlinear nature of dynamic systems. Analytical solution of probability function from master equation or Fokker-Planck equation exists only for some very simple cases.[14,15] Most of the studies are concentrated on the steady state (SS) solutions which enable us to probe the multistability and the passage between them. Transient processes preceed the final SS are considered as crucial to the nonequiiibrium systems. Approximation schemes[16,17] are always required in order to have a thorough study. Detailed analysis in some one-variable systems has successed in providing a stochastic picture of noise-induced transition.'

We extend the moment expansion scheme[17] to the bivariate system in this report. A famous Lotka-Volterra model[18,19] is chosen for study. In next section we describe this model and its deterministic transient properties. In Sec.III, the moment expansion scheme will be employed to derive the internal stochasticity from the master equation. In Sec.IV, all possible white noises will be considered and moment equations will be derived. Analytical and numerical results of noise effects will be presented in Sec.V, followed by discussions in Sec.VI.

II. MODEL SYSTEM AND DETERMINISTIC TRANSIENT

The model system is originated from a set of coupled autocatalytic chemical reaction which can be expressed by,[18]

$$\begin{aligned} A + X &\xrightarrow{c_1} 2X \\ X + Y &\xrightarrow{c_2} 2Y \\ Y &\xrightarrow{c_3} B \end{aligned} \tag{1}$$

where c_i's are reaction rate constants. External agent is monitoring the reaction system constantly to guarantee a constant supply of the reactant A, and to pump out the final products B. Stirring mechanism is introduced to render spatial homogeneity over the system. By denoting the concentrations of X and Y by x and y, the reaction can be described by a set of coupled differential equations,

$$\frac{d}{dt}x = k_1 x - \beta x y$$
$$\frac{d}{dt}y = -k_2 y + \beta x y \tag{2}$$

where,

$$k_1 = c_1 A, \quad k_2 = c_3, \quad \beta = c_2. \tag{3}$$

The rate Eqs. (2) are identical to those for a predator-prey model[19] in ecological system in which species Y is predating on X. Variations of the form, and extensions to higher dimensions do exist.[19] Modified version in multi-dimensions has also been used to model more complex system of neural networks?' In this report, we shall investigate the reaction system described by Eqs. (2) only.

It is easy to see that these two equations allow two SS solutions. The trivial solution with

$$x_s = y_s = 0 \tag{4}$$

represents a reaction being extinct. Deterministically, this unique state is possible either right from the beginning if we start with initial concentrations $x(0) = y(0) = 0$, or after decaying of Y if $x(0) = 0$ and $y(0) \neq 0$. This SS is unstable since the maximum eigenvalue of linear stability analysis[4] equals to k_1 which is assumed to be positive. This SS is also called absorbing SS, since once this state is reached there will be no recovery. The second class of SS is also a special one,

$$x_s = k_2/\beta, \quad y_s = k_1/\beta. \tag{5}$$

It represents coexistance of X and Y, and can only be set up right from beginning if $x(0) = x_s$ and $y(0) = y_s$. This SS is marginally stable since the corresponding eigenvalues have zero real value,

$$\lambda = \pm i\sqrt{k_1 k_2}. \tag{6}$$

For any arbitrary initial state, x(t) and y(t) will oscillate about their coexistance SS values defined by Eq. (5). Transient behavior of the reacting system can be described by a closed path in the phase space (x,y). Path will be deformed from circular shape if the initial point (x_0, y_0) is not close to the SS point.

Closed trajectory in phase space implies that x(t) and y(t) are both oscillating with same frequency. The period is also found to be proportion to $\sqrt{k_1 k_2}$ even for deformed trajectories, and is found to be independent to β. It is interesting to find that this p-independency retains even with noise effects.

There is another invariance for this oscillating system. A Lyapunov function defined by,[21]

$$K(x, y) = x - x_s \ln(x/x_s) + Y - y_s \ln(y/y_s), \tag{7}$$

can be shown to be positive definite and time independent. In the above, and in all what follows, we use (x_s, y_s) to denote the coexistent SS values given by Eq. (5).

III. INTRINSIC STOCHASTICITY AND MOMENT EQUATIONS

In this section we present the stochastic properties inherent to the system before we start the study of noise effects. Stochastic consideration is necessary since that all the reactions are probabilistic in nature and that reaction rate parameters are themselves averaged quantities. By treating reaction processes as single-step birth-death processes, transient properties can be described by the probability function P(x,y;t) which satisfies a master equation,[14]

$$\begin{aligned}\frac{\partial P(x,y;t)}{\partial t} = \; & k_1(x-1)P(x-1,y;t) + \beta(x+1)(y-1)P(x+1,y-1;t) \\ & + k_2(y+1)P(x,y+1;t) - (k_1 x + k_2 y + \beta xy)P(x,y;t).\end{aligned} \tag{8}$$

Analytical solution of this differential difference equation is impossible. Only SS solution is known to be the extinction state.[22] Most of the characteristics of P(x,y;t) are described by the first few moments which are defined by,

$$\overline{x^m y^n} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} x^m y^n P(x,y;t). \tag{9}$$

Moment equations can be derived from Eq. (8), some of them are given by,

$$\begin{aligned}\dot{\overline{x}} &= k_1\overline{x} - \beta\overline{xy} \\ \dot{\overline{y}} &= -k_1\overline{y} + \beta\overline{xy} \\ \dot{\overline{x^2}} &= 2k_1\overline{x^2} + k_1\overline{x} - 2\beta\overline{x^2y} + \beta\overline{xy} \\ \dot{\overline{y^2}} &= -2k_2\overline{x^2} + k_2\overline{x} + 2\beta\overline{xy^2} + \beta\overline{xy} \\ \dot{\overline{xy}} &= (k_1 - k_2 - \beta)\overline{xy} + \beta(\overline{x^2y} - \overline{xy^2}).\end{aligned} \tag{10}$$

In the above the dot represents time derivative.

It is evident from the quadratic nonlinearity of the rate Eq. (2) that moment equation of any order will always couple to those of higer order. Moment expansion approximation[17] which was quite successive in treating one dimensional systems can be extended to truncate the hierarchy of coupled equations up to second order. This approximation enable us to express the third moments in terms of those of first and second orders,

$$\overline{xyz} \approx \overline{xy}\bar{z} + \overline{yz}\bar{x} + \bar{z}\overline{xy} - 2\bar{x}\bar{y}\bar{z} \tag{11}$$

This scheme is valid for fluctuations not too large, since we keep the first and second order of deviation and drop those of third and higher orders. After some mathematics, we get a set of five coupled moment equations in closed form,

$$\begin{aligned}
\dot{\overline{x}} &= k_1\overline{x} - \beta\bar{x}\bar{y} - \beta\sigma_{xy} \\
\dot{\overline{y}} &= -k_2\overline{y} + \beta\bar{x}\bar{y} + \beta\sigma_{xy} \\
\dot{\sigma}_x &= 2(k_1 - \beta\overline{y})\sigma_x - 2\beta\overline{x}\sigma_{xy} + 2k_1\overline{x} - \dot{\overline{x}} \\
\dot{\sigma}_y &= -2(k_2 - \beta\overline{x})\sigma_y + 2\beta\overline{y}\sigma_{xy} + 2k_2\overline{y} + \dot{\overline{y}} \\
\dot{\sigma}_{xy} &= [k_1 - k_2 + \beta(\overline{x} - \overline{y} - 1)]\sigma_{xy} + \beta(\overline{y}\sigma_x - \overline{x}\sigma_y) + \beta\bar{x}\bar{y},
\end{aligned} \tag{12}$$

where, $\overline{x}$ and $\overline{y}$ re averaged concentrations and σ_{ij}'s are variances defined by,

$$\sigma_x = \overline{x^2} - \overline{x}^2, \quad \sigma_y = \overline{y^2} - \overline{y}^2, \quad \sigma_{xy} = \overline{xy} - \bar{x}\bar{y}. \tag{13}$$

Now the time evolution of averaged concentrations depends on that of convariance σ_{xy}. This coupling brings the influence of increasing fluctuations into the reacting system, and drive the system from marginal stability to instability.

Numerical results show that both $\overline{x}$ and $\overline{y}$ are oscillating with similar frequency which is slightly different from the deterministic one. The cyclic trajectory in phase space defined by $(\overline{x},\overline{y})$ is no longer closed, instead it spirals toward the absorbing boundary. We extend the definition of Lyapunov function $K(x,y)$ to a stochastic one,

$$K(\overline{x}, \overline{y}) = \overline{x} - x_s \ln(\overline{x}/x_s) + \overline{y} - y_s \ln(\overline{y}/y_s). \tag{14}$$

It can be shown that this function is no longer an invariant, and that its time-rate of change is given by,

$$\dot{K}(\overline{x}, \overline{y}) = \beta(x_s/\overline{x} - y_s/\overline{y})\sigma_{xy}. \tag{15}$$

Numerical result shows that $K(\overline{x},\overline{y})$ is oscillating and is always larger than the deterministic fixed value. This oscillatory behavior is plotted together with those of concentrations in Fig. 1, in which we choose the deterministic fixed point as the initial state, and define a set of reduced variables,

$$\zeta_x = \frac{\overline{x} - x_s}{x_s}, \quad \zeta_y = \frac{\overline{y} - y_s}{y_s}, \tag{16}$$

In deterministic limit all three curves would fall on a common horizontal line. We see that in-

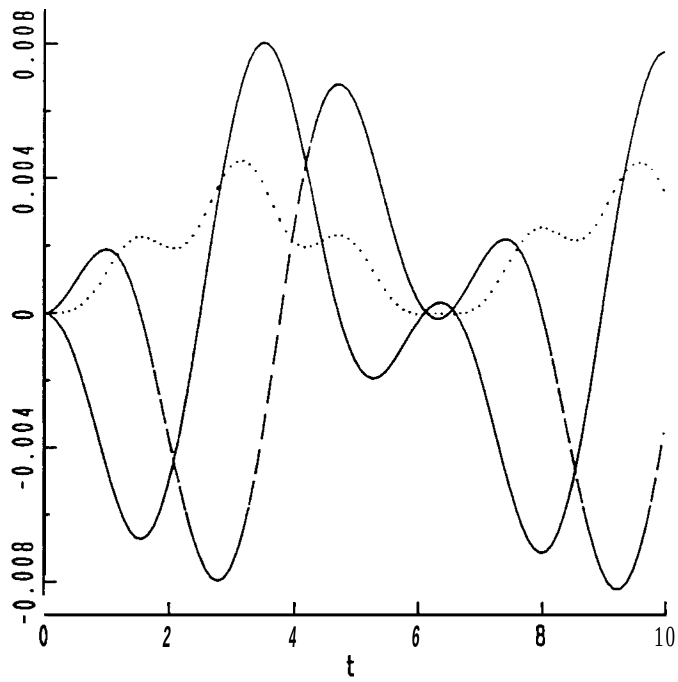

FIG. 1. Deviations in concentrations from the deterministic SS values are plotted as function of time. Solid and dashed curves stand for deviations in $\bar{x}$ and $\bar{y}$ respectively. Dotted curve stands for the deviation of $K(\bar{x},\bar{y})$ from the deterministic SS value. Intrinsic fluctuations are considered and $k_1 = k_2 = 1$ and $\beta = 0.01$ are used.

trinsic fluctuations drive the deterministic SS point to an oscillation. We also see that the oscillation is not perfectly periodic, and that $K(\bar{x},\bar{y})$ is oscillating roughly with the same frequency of cyclic trajectory.

Fig. 2 displays the behavior of relative fluctuations which are defined by,

$$R_x = \sqrt{\sigma_x/\bar{x}^2}, \; R_y = \sqrt{\sigma_y/\bar{y}^2}. \tag{17}$$

It is interesting to find that both are monotonically increasing with oscillatory patterns, and that fluctuation in $\bar{y}$ is always larger than that of $\bar{x}$. We have to note here that the parameter β which couples fluctuations to the means in Eq. (12), play a significant role in determining the values of relative fluctuations.

IV. WHITE NOISE FORMULATION

The deterministic Eq. (2) of reacting system contain three rate parameters, each of which might subject to noises. Though we are going to treat them as decoupled and to descuss their

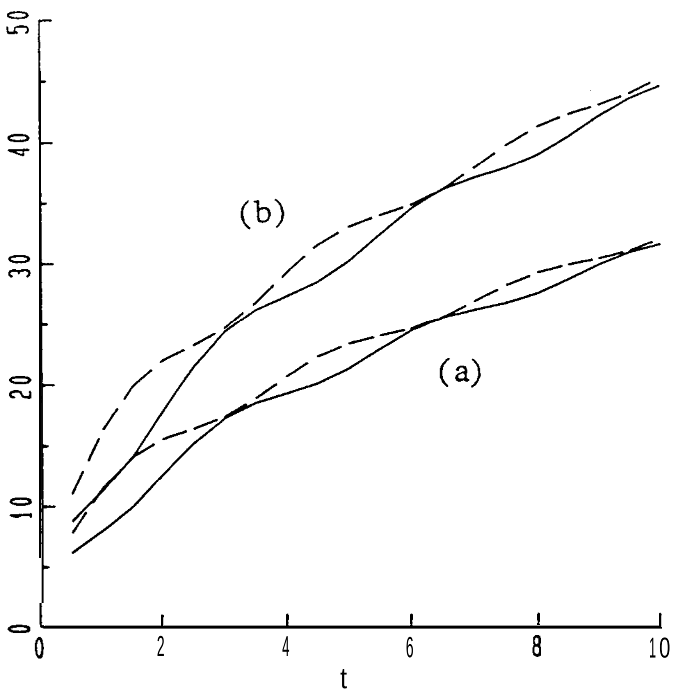

FIG. 2. Relative fluctuations (in %) of $\bar{x}$ (solid curves) and $\bar{y}$ (dashed curved) are shown to increase with time. Internal fluctuations are considered and the system starts with deterministic fixed point. We use $k_1 = k_2 = 1$, (a) $\beta = 0.005$ and (b) $\beta = 0.01$.

effects separately, the stochastic treatment will be formulated in a general way so that each parameter $\lambda_{i,t}$ is fluctuating randomly about its means λ_i and with amplitude D_i,

$$\lambda_{i,t} = \lambda_i + D_i \xi_i(t), \tag{18}$$

where $\xi_i(t)$ is a Guassian process which has zero mean and is b-correlated,

$$< \xi_i(t) >= 0, \quad < \xi_i(t)\xi_j(t') >= \delta_{ij}\delta(t - t'). \tag{19}$$

We assume that there is no correlation among noises, though some of them might be contributed by same source of environmental fluctuations. The white noise realization is basing on the assumption that all noises are fluctuating much faster than reaction rate. Concentrations x and y are now considered as stochastic variables and satisfy the following stochastic equations,

$$\begin{aligned} \dot{x} &= h_x(x, y) + \sum_i \lambda_i g_{xi}(x, y)\xi_i(t) \\ \dot{y} &= h_y(x, y) + \sum \lambda_j g_{yj}(x, y)\xi_j(t) \end{aligned} \tag{20}$$

In the above g_{xi} and g_{yi} are those functions multiplied to noise parameter λ_i which appears in the x- and y- equations respectively. Summations are carried out to include all noises, h_x and h_y are those parts not affected by noises. For different noises all h-and g- functions might be different. We note that if we consider all possible noises at the same time, h_x and h_y will both be zero.

Treatment with different stochastic culculus will lead to different form of Fokker-Planck[15] equation which governs the time-evolution of probability function. For the two most quoted cases, we have a two dimensional Fokker-Planck equation in a general form,

$$\begin{aligned}\frac{\partial}{\partial t}P(x,y;t) = & -\frac{\partial}{\partial x}\left[(f_x + \frac{\nu-1}{2}\sum_i D_i^2 g_{xi}\frac{\partial}{\partial x}g_{xi})P\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_i D_i^2 g_{xi}^2 P \\ & -\frac{\partial}{\partial y}\left[(f_y + \frac{\nu-1}{2}\sum_j D_j^2 g_{yi}\frac{\partial}{\partial y}g_{yi})P\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sum_j D_j^2 g_{yj}^2 P.\end{aligned} \tag{21}$$

where $\nu = 1,2$ stand respectively for the results in Ito and Stratonovich calculi. The function f_x and f_y are those on the right hand sides of deterministic x- and y- equations respectively. Again analytic solution for $P(x,y;t)$ is unlikely, we look for the moments defined by,

$$\overline{x^m y^n} = \int_0^\infty \int_0^\infty x^m y^n P(x,y;t)dxdy. \tag{22}$$

Moment equations can be derived and expressed as,

$$\begin{aligned}\frac{d}{dt}\overline{x^m y^n} = \; & m < x^{m-1}y^n f_x > +n < x^m y^{n-1} f_y > \\ & +\frac{1}{2}\left[m(m-1)\sum_i D_i^2 < x^{m-2}y^n g_{xi}^2 > \right. \\ & \left. +n(n-1)\sum_j D_j^2 < x^m y^{n-2} g_{yj}^2 > \right] \\ & +\frac{\nu-1}{2}\left[m\sum_i D_j^2 < x^{m-1}y^n g_{xi}\frac{\partial}{\partial x}g_{xi} > \right. \\ & \left. +n\sum_j D_j^2 < x^m y^{n-1} g_{yj}\frac{\partial}{\partial y}g_{yj} > \right]\end{aligned} \tag{23}$$

where we use both over-head bar and bracket $<\ldots>$ to denote averages. We express the first five moment equations in the following compact form,

$$
\begin{aligned}
\overline{x}_u &= \; < f_u > +(\nu-1)\sum_i D_i^2 < g_{ui}\frac{\partial g_{ui}}{\partial x_i} >, \\
\dot{\sigma}_{uv} &= \; < x_u f_v > - \overline{x}_u < f_v > + < x_v f_u > - \overline{x}_v < f_u > + \sum_i D_i^2 < g_{ui}^2 > \delta_{uv} \\
&+ \frac{\nu-1}{2}\left\{ \sum_i D_i^2 \left[< x_v g_{ui}\frac{\partial g_{ui}}{\partial x_u} > - \overline{x}_v < g_{ui}\frac{\partial g_{ui}}{\partial x_u} > \right] \right. \\
&\left. + \sum_j D_j^2 \left[< x_u g_{vj}\frac{\partial g_{vi}}{\partial x_v} > - \overline{x}_u < g_{vj}\frac{\partial g_{vj}}{\partial x_v} > \right] \right\}.
\end{aligned}
\tag{24}
$$

In the above, subscripts (u,v) stands for (x,y), while x_u = x, and x_v = y. Again summations over i and j are for those noises considered in x- and y- equations respectively.

Different noises have different forms of g_{ki}, and contribute different orders of nonlinearity to the moment Eq. (24). These five equations do look complicated, since they include all possible noises. For a specific study of a given noise or a small set of noises, we can simplify these equations by dropping out those noise intensities D^2 which are not under consideration. As for the case of internal fluctuation, approximation scheme has to be employed in order to render these equations into closed form.

V. NOISE EFFECTS

In order to study noise effects systematically and to elucidate the role played by each rate parameter in stochastic transient, we shall investigate each type of noise separately.

The simplest noise is the one in which environmental fluctuations contribute state-independent terms to the rate equations. These terms affect the time- evolution of concentrations as additional forces which fluctuate randomly about zero with amplitude D. Now with $g_x = g_y = 1$ and the approximation Eq. (11), the moment Eq. (24) can be simplified and expressed in a handy form,

$$
\begin{aligned}
\dot{\overline{x}} &= (k_1 - \beta\overline{y})\overline{x} - \beta\sigma_{xy} + S_1 \\
\dot{\overline{y}} &= (-k_2 + \beta\overline{x})\overline{y} + \beta\sigma_{xy} + S_2 \\
\dot{\sigma}_x &= 2(k_1 - \beta\overline{y})\sigma_x - 2\beta\overline{x}\sigma_{xy} + S_3 \\
\dot{\sigma}_y &= 2(-k_2 + \beta\overline{x})\sigma_y + 2\beta\overline{y}\sigma_{xy} + S_4 \\
\dot{\sigma}_{xy} &= (k_1 - \beta\overline{y} - k_2 + \beta\overline{x})\sigma_{xy} + \beta(\overline{y}\sigma_x - \text{L-a,)} + S_5
\end{aligned}
\tag{25}
$$

where,

$$
S_1 = S_2 = S_5 = 0, \quad S_3 = D_1^2, \qquad S_4 = D_2^2. \tag{26}
$$

It is interesting to note that noise intensities do not appear explicitly in $\bar{x}$- and $\bar{y}$- equations. Instead they appear as net increasing rates in σ_x- and σ_y- equations. The noise-assisted fluctuations affect mean concentrations indirectly through the coupling of σ_{xy}. Again β plays a crucial role in this coupling. Numerical results show that additive noises are less significant than multiplicative ones.

Effect of additive noises is demonstrated in Fig. 3, in which a deterministic fixed point is drifted to a spiraling oscillation. Results with various combinations of noise intensities are presented for comparision. Instead of plotting the oscillating concentrations against time, variables ζ_x and ζ_y defined in Eq. (16) are used to construct a phase space. As for the case of internal fluctuation, additive noises are found to induce fluctuation catastrophe in such a way that the phase trajectories will soon spiraling toward aborbing boundaries. It is interesting to find that $\bar{y}$ is more vulnerable to noise effects.

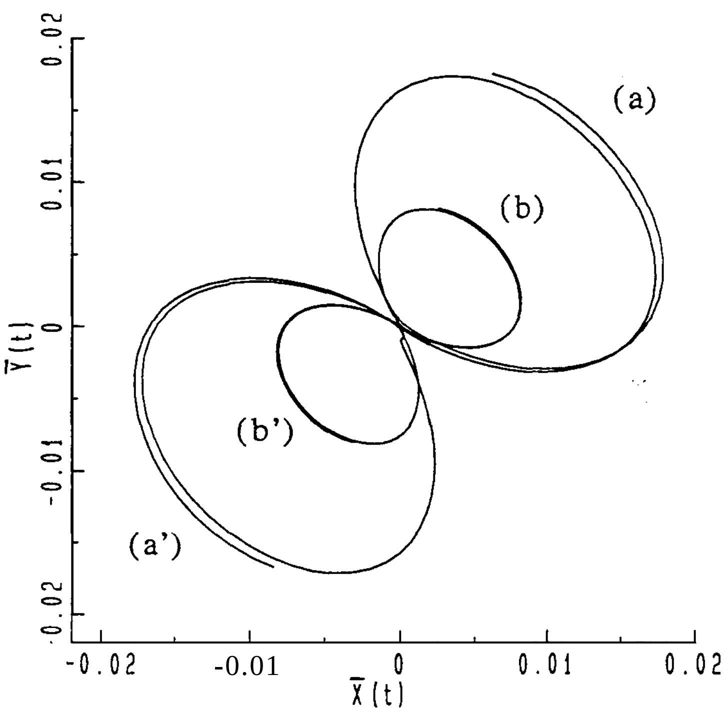

FIG. 3. Additive noises drive a deterministic SS point to spiraling phase trajectories. $k_1 = k_2 = 1$ and $\beta = 0.01$ are used. Four combinations of noise intensities (D_1^2, D_2^2) are chosen: (a) (10,200); (b) (10,100); (a') (200,10); and (b') (100,10).

The stochastic K-function defined by Eq. (14) is found to be oscillating with time in a similar fashion of Fig. 1. In fact the time rate of change of this function is found to be precisely the same form of Eq. (15) for the internal fluctuations.

The multiplicative noise of first order can be introduced by either one, or both of the fluctuating parameters k_1 and k_2. With $g_x = x$ and $g_y = -y$, the moment Eq. (24) can be expanded into a set of five approximate equations as expressed by Eq. (25), with,

$$S_1 = \frac{\nu-1}{2} D_{k_1}^2 \overline{x}, \qquad S_2 = \frac{\nu-1}{2} D_{k_2}^2 \overline{y}, \quad S_3 = D_{k_1}^2(\nu\sigma_x + \overline{x}^2), \quad S_4 = D_{k_2}^2(\nu\sigma_y + \overline{y}^2), \quad S_5 = \frac{\nu-1}{2}(D_{k_1}^2 + D_{k_2}^2). \tag{27}$$

We see that, noise intensities enter all moment equations in multiplicative ways, and that results based on Stratonovich interpretation (ν = 2) show more noise influences.

The change of the stochastic K-function can be calculated analytically with the help of moment equations. Final result can be expressed in an illustrative form,

$$\dot{K} = (\dot{K})_{if} + \frac{\nu-1}{2}\left[D_{k_1}^2(\overline{x} - x_s) + D_{k_2}^2(\overline{y} - y_s)\right], \tag{28}$$

where $(\dot{K})_{if}$ is the part due to internal fluctuations, as given by Eq. (15). The contribution by noises can easily be identified, and is found to be zero in Ito interpretation (ν = 1). Fig. 4 shows the oscillatory patterns due to noises in k_1 and k_2 individually.

The noise in β would produce much more complicated result since β appears in both rate Eq. (2) in a quadratic multiplicative manner. For $g_x = -xy$ and $g_y = xy$, the additional terms in the moment Eq. (25) are now given by,

$$\begin{aligned}
S_1 &= \frac{\nu-1}{2} D_\beta^2 \left[\overline{x}(\sigma_y + \overline{y}^2) + 2\overline{y}\sigma_{xy}\right] \\
S_2 &= \frac{\nu-1}{2} D_\beta^2 \left[\overline{y}(\sigma_x + \overline{x}^2) + 2\overline{x}\sigma_{xy}\right] \\
S_3 &= D_\beta^2 \left[\nu\overline{y}^2\sigma_x + 2(\nu+1)\bar{x}\bar{y}\sigma_{xy} + \overline{x}^2(\overline{y}^2 + \sigma_y)\right] \\
S_4 &= D_\beta^2 \left[\nu\overline{x}^2\sigma_y + 2(\nu+1)\bar{x}\bar{y}\sigma_{xy} + \overline{y}^2(\overline{x}^2 + \sigma_x)\right] \\
S_5 &= \frac{\nu-1}{2} D_\beta^2 \left[(\overline{x}^2 + \sigma_y^2)\sigma_{xy} + 2\bar{x}\bar{y}(\sigma_x + \sigma_y)\right].
\end{aligned} \tag{29}$$

In the above, due to higher order of nonlinearities, we use approximate Eq. (11) and a higher order of moment expansion relationship,

$$< xysu > \approx \bar{x}\overline{y}\overline{z}\overline{u} + \bar{x}\bar{y}\sigma_{zu} + \bar{y}\bar{z}\sigma_{ux} + \bar{z}\bar{u}\sigma_{xy} + \bar{u}\bar{x}\sigma_{yz} + \bar{x}\bar{z}\sigma_{yu} + \bar{y}\bar{u}\sigma_{xz}. \tag{30}$$

The noise effect is more significant for this quadratic multiplicative noise, as we can see from Fig. 4 that the invariant function is now oscillating with much larger amplitude than other cases. The corresponding changes in $K(\bar{x},\bar{y})$ is calculated to be,

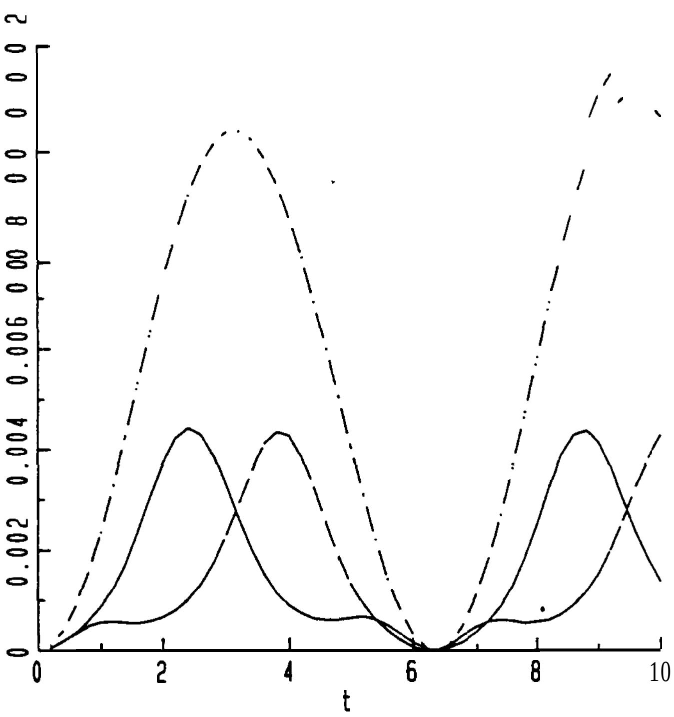

FIG. 4. Multiplicative noises drive the deterministic SS value of K-function to oscillate with time. $k_1 = k_2 = 1$ and $\beta = 0.01$ are used. Solid, dashed and dash-dotted curves stand for the changes of K-function due to noises in k_1, k_2 and β respectively. A common value of relative noise intensity, defined by Eq. (32), R = 0.1 is used.

$$\begin{aligned} \dot{K} = {} & (\dot{K})_{if} + (\nu - 1) D_\beta^2 \left[\frac{\overline{y}}{\overline{x}} \sigma_{xy} + \frac{1}{2}(\sigma_y + \overline{y}^2)(\overline{y} - y_s) \right. \\ & \left. + \frac{\overline{x}}{\overline{y}} \sigma_{xy} + \frac{1}{2}(\sigma_x + \overline{y}^2)(\overline{x} - x_s) \right]. \end{aligned} \tag{31}$$

Again contribution to the change of $K(\overline{x},\overline{y})$ can be identified to be associated with different sources of stochasticities. From Fig. 4, we also note that each $K(x,y)$ due to individual noise is oscillating with almost the same frequency of deterministic transient and that D_β^2 causes largest deviation in $K(\overline{x},\overline{y})$.

Success in truncating the moment equations into a closed form enable us to probe transient properties in more detail. From the numerical solutions of $\overline{x}(t)$ and y(t), we can estimate the approximate period of spiraling trajectories in the stochastic phase space. Results are plotted in Fig. 5 for all multiplicative noises. These curves can be used to demonstrate a noise- induced

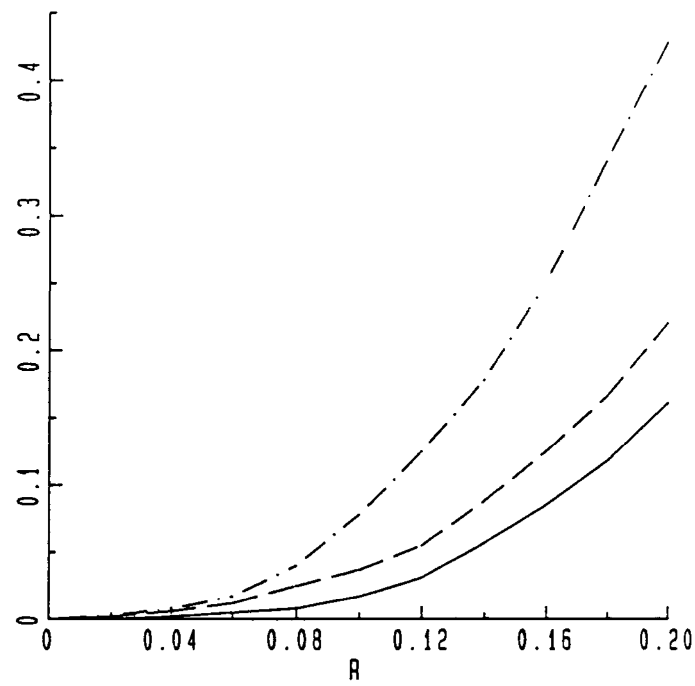

FIG. 5. Period of oscillating transient is plotted against the relative noise intensity R, which is defined by Eq. (32). Initial state $(x_0, y_0) = 1.01\ (x_s, y_s)$ is assumed; $k_1 = k_2 = 1$ and $\beta = 0.01$ are used. Solid, dashed and dash-dotted curves stand for the increases of period due to noises in k_1, k_2 and β respectively.

slowing-down phenomenon. Since we choose a common set of average values for rate parameters and use same initial conditions, these curves can be used to compare the relative significances of effects due to different noises. In Fig. 5, we use a relative noise intensity defined by,

$$R_i = \sqrt{D_i^2}/\lambda_i, \tag{32}$$

where $\lambda_i = (k_1, k_2, \beta)$. It is found that noise effect is most significant for β, and that the system is least vulnerable to the noise in k_1. All the periods are estimated for the first cycle only. It appears that periods tend to increase with time since subsequent cycles take longer and longer period. A systematic and quantitative study of this tendency, in a fashion similar to Fig. 5, is not feasible since fluctuations are also increasing with time.

VI. DISCUSSIONS AND CONCLUSIONS

With the help of relationships derived from moment expansion scheme, moment equations can be truncated into five and expressed in a close form. Detailed analysis enable us to study stochastic characteristics of transient processes, to estimate approximate period, and to define a stochastic K-function for the unstable system. Both period increases and K-function deviations demonstrate quantitatively the noise effect on reacting system. Furthermore, we derive expressions for the time-rate of change of this K-function, which indicate explicitly distinct contributions from various types of noises.

Results from different stochastic calculi are qualitatively similar. Only those derived from Stratonovich interpretation are shown together with a common set of average parameters $k_1 = k_2 = 1$ and $\beta = 0.01$, Transient properties due to external noises are found to be similar to those due to internal fluctuations. Phase trajectories always spiral toward extinction while fluctuations increase monotonically with time. In general, larger noise will result in larger concentration fluctuations and higher instability. Fluctuation catastrophe is always a destination for this reaction process, even if it starts with a deterministic fixed point.

The possibe fate of having explosion in concetration x(t), discussed by Arnold, Horsthemke and Stucki,[21] is found to be likely in stochastic sense. Since relative fluctuation in y(t) is always larger than that of x(t), species Y is more vulnerable to fluctuation catastrophe. This possibility will be enhanced by the following two factors. The first factor is size effect, ' ie, if the system starts with small concentration in Y. The second factor is a noise effect due to fluctuating β, since growth of Y counts purely on this parameter. This result seems to contradict the situations in chemical and ecological systems which do show oscillations sustained over long periods of time. Keizer suggested that[23] the system will be stabilized if we consider that the reactant A in Eq. (1) is not kept at fixed, but is perturbed by an additive white noise.

Results show consistently that the system is most susceptible to the noise effect due to fluctuating/?. This parameter, though play no role in determining the period, produces largest noise effect in all aspects. Through its presence in many parts of moment equations, β plays a most important role in stochastic properties of the system. It couples fluctuations to average concentrations. Relative fluctuations in concentrations are found to be determined predominantly by β. In this model, the multiplicative function of β is a nonlinear function while those of k_1 and k_2 are linear. All these tend to suggest that non-linearity amplifies noise effects. It is interesting to note here that in one dimensional systems cubic model is more vulnerable to noise effects than the quadratic one.[24]

Noise-induced *critical* slowing-down has been investigated in several one-dimensional systems with stable SS.[9,24-27] The Lotka-Volterra model is known to have oscillatory transient with no stable SS. We manage to demonstrate quantitatively a noise-induced slowing-down phenomenon by estimating the approximate period for the stochastic transient.

All results are subject to errors inherent to the moment expansion scheme. In all the computations, we start with initial states near the deterministic SS so that relative fluctuations are insignificant and that our approximation remains to be reliable.

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