CHINESE JOURNAL OF PHYSICS VOL. 25, NO. 2 SUMMER 1987

A Universal Formula of Bäcklund Transformation Equations

H. C. Yen (顏晃徹)
Department of Physics, National Tsing Hua University, Hsinchu. Taiwan 30043
(Received 24 February, 1987)

We present a unified derivation of Backlund transformations (BT ' s) and exhibit a universal formula for such BT's, for a broad class of two-dimensional nonlinear evolution equations.

I. INTRODUCTION

The Backlund transformations (BT ' s) for some two-dimensional nonlinear evolution equations, such as the sine-Gordon equation and the Kortweg-de Vries equation, have been well known for a long time[1], and several somewhat unified derivations of such BT's have been discussed, for example, by Chen[2] and by Konno and Wadati[3]. These derivations basically consist of converting the linear system associated with the nonlinear equation into a pair of coupled Riccati equations, and looking for connections between two possible solutions of the nonlinear equation through symmetry properties of the Riccati equations.

In this paper we present an alternative derivation of the BT's, which essentially consists of taking a very simple ansatz for the spectral dependence of the Darboux matrix connecting two solutions of the linear system, as proposed by Levi, Ragnisco and Sym[4], and by Chau[5]. This provides not only another unified derivation of the BT's for a large class of nonlinear evolution equations, but also a universal formula for the BT's so obtained. Thus the main result of this paper is contained in the explicit universal formula for the BT's, Eqs. (3.22).

II. A CLASS OF NONLINEAR EVOLUTION EQUATIONS

We will be concerned with the class of nonlinear evolution equations which can be obtained from the integrability condition of the following particular type of coupled linear partial differential equations[1]:

$$\psi_x = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \psi \qquad \psi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \psi \tag{2.1a,b}$$

where λ is a constant parameter, q and r are X-independent functions of the fundamental field u(x, t) and its various x-derivatives, while A, B and C are h-dependent functionals of u and its x-derivatives. The function ψ can be taken as either a 2 x 2 matrix, or a two-component column vector.

The above class of equations includes

1. K-dV eq. $$u_t + u_{xxx} \mp 6u\, u_x = 0 \ , \tag{2.2a}$$

with $q = u,\ r = \pm 1,\quad A = \pm u_x \mp 2i\lambda u - 4i\lambda^3 \ ,$

$$B = - u_{,,} + 2i\lambda u_x + 4\lambda^2 u \pm 2u^2 \ , \tag{2.2b}$$

$C = 2u \pm 4\lambda^2$

2. mK-dV eq. $$u_t + u_{xxx} \mp 6u^2 u_x = 0 \ , \tag{2.3a}$$

with $q = u,\ \ r = \pm\ q,\ \ A = \mp\ 2i\lambda u^2 - 4i\lambda^3 \ ,$

$$B = -u_{xx} + 2i\lambda u_x + 4\lambda^2 u \pm 2u^3 \tag{2.3b}$$

$C = \mp u_{xx} \mp 2i\lambda u_x \pm 4\lambda^2 u + 2u^3 \ ;$

3. cubic Schrodinger eq. $$iu_{,} + u_{xx} \mp 2\,|\,u\,|^2\, u = 0 \tag{2.4a}$$

with $q = u,\quad r = \pm\, q^*,\quad A = \mp\, i\,|\,u\,|^2 - 2i\lambda^2 \ ,$

$$B = iu_x + 2\lambda u, \qquad C = \pm B^* \ ; \tag{2.4b}$$

4. sine-Gordon eq. $$u_{xt} = \sin u, \tag{2.5a}$$

with $$q = -r = \frac{1}{2} u_x,\ \ A = \frac{i}{4\lambda} \cos\, u,\ \ B = C = \frac{-i}{4\lambda} \sin\, u \ ; \tag{2.5b}$$

5. Liouville eq. $$u_{xt} = 2\, e^u \ , \tag{2.6a}$$

with $$q = r = \frac{1}{2}\, u_x,\ \ A = \frac{i}{2\lambda}\, e^u,\quad B = \_C = \frac{-i}{2\lambda}\, e^u \tag{2.6b}$$

In this paper, we will restrict ourselves only to three kinds of r:

$$r = \pm 1, \qquad r = \pm q, \qquad r = \pm q^*, \tag{2.7}$$

as exemplified by Eqs. (2.2) – (2.6). The integrability of (2.1) implies

$$0 = A_x - qC + rB \tag{2.8a}$$

$$q_t = B_x + 2qA + 2i\lambda B \tag{2.8b}$$

$$r_t = C_x - 2rA - 2i\lambda C \tag{2.8c}$$

III. A UNIFIED DERIVATION OF THE BT

Consider two different solutions $u(x, t)$ and $u'(x, t)$ to a particular nonlinear evolution equation. Corresponding to them we have two different linear systems $\psi(\lambda; x, t)$ and $\psi'(\lambda; x, t)$ satisfying Eqs. (2.1). Suppose ψ and ψ' are related by

$$\psi'(\lambda; x, t) = R(\lambda; x, t)\psi(\lambda; x, t) \tag{3.1}$$

where the 2 x 2 Darboux matrix R is assumed to have a very simple λ-dependence:

$$R(\lambda; x, t) = R_0(x, t) + \lambda R_1(x, t) \tag{3.2}$$

This ansatz (3.2) has been proposed and extensively discussed in Ref. 4, and has been successfully used to derive BT's for the 2-D principal chiral fields and the 4-D self-dual Yang-Mills fields[5]. From (2.1) and (3.1), we get a pair of linear equations in R:

$$R_x = \begin{pmatrix} -i\lambda & q' \\ r' & i\lambda \end{pmatrix} R - R \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix}, \tag{3.3a}$$

$$R_t = \begin{pmatrix} A' & B' \\ C' & -A' \end{pmatrix} R - R \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{3.3b}$$

Formally Eqs. (3.3) consist of eight separate equations, but not all of them are necessarily independent. We now try to reduce them down to two final independent equations by suitably expressing R completely in terms of u and u'. The last two coupled partial differential equations relating u and u' will then be the desired BT equations.

From (3.2) and (3.3a), we immediately get

$$0 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} R_1 - R_1 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and two obvious simplest solutions of R_1 are

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

In the following we will take the first choice in (3.4). A derivation of BT with the second choice of R_1 can of course be similarly worked out. Let

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.5}$$

with a, b, c and d to be determined from (3.3). Surprisingly, however, R_0 can in fact be completely determined by (3.3a) alone, with (3.3b) providing the desired BT eqs., as will be seen shortly.

Substituting (3.5) into (3.3a), we get

$$b = \frac{1}{2}(q' + q) \qquad c = \frac{1}{2}(r' + r), \tag{3.6}$$

$$b_x = q'd - qa, \qquad c_x = r'a - rd, \tag{3.7}$$

and

$$a_x = q'c - rb \qquad d_x = r'b - qc, \tag{3.8}$$

which then combine to give

$$a = \frac{i(q'r'_x + q'r_x + q'_x r + q_x r)}{2(q'r' - qr)}, \qquad d = \frac{i(r'q'_x + r'q_x + r'_x q + r_x q)}{2(q'r' - qr)} \tag{3.9}$$

and

$$a_x = d_x = \frac{1}{2}(q'r' - qr). \tag{3.10}$$

Eq. (3.10) suggests that we try a = d, which implies, by (3.9) that

$$(q'+q)_x(r'-r) = (r'+r)_x(q'-q) \quad . \tag{3.11}$$

This last condition is automatically satisfied for the cases $r = \pm 1$ and $r = \pm q$, and can be satisfied for the case $r = \pm q^*$ if we assume $(q'+q)_x/(q'-q)$ to be real, which is indeed a consistent assumption,-as can be checked later on.

Thus from (3.9) we have

$$a = d = \frac{a+d}{2} = \frac{i[(q'+q)(r'+r)]_x}{4(q'r'-qr)} \tag{3.12}$$

which can be simplified by using (3.11) to become

$$a = \frac{i}{2}\frac{(q'+q)_x}{(q'-q)} \tag{3.13}$$

So the final form of R is surprisingly simple:

$$R = \begin{pmatrix} \lambda + a & b \\ c & -\lambda + d \end{pmatrix} = \begin{pmatrix} \lambda + \dfrac{i(q'+q)_x}{2(q'-q)} & \dfrac{i}{2}(q'+q) \\ \dfrac{i}{2}(r'+r) & -\lambda + \dfrac{i(q'+q)_x}{2(q'-q)} \end{pmatrix} \tag{3.14}$$

So far we-have not yet used (3.3b) in computing R, and we are not quite done with Eq. (3.10). Substituting (3.14) into (3.3b), we get

$$b_t = -\lambda(B'+B) + a(B'-B) + b(A'+A) \tag{3.15}$$

$$c_t = \lambda(C'+C) + a(C'-C) - c(A'+A) \quad , \tag{3.16}$$

$$0 = 2a(A'-A) - b(C'+C) + c(B'+B) \tag{3.17}$$

$$a_t = \lambda(A'-A) + \frac{1}{2}c(B'-B) + \frac{1}{2}b(C'-C) \tag{3.18}$$

From the above four equations, it is easy to prove that

$$(bc - a^2)_t = 0 \quad , \tag{3.19}$$

while (3.7) and (3.8) implies (bc-ad), = 0, which becomes $(bc-a^2)_x = 0$ since a = d. Therefore $bc-a^2$ is constant, or more explicitly,

$$\left[\frac{(q'+q),}{(q'-q)}\right]^2 - (q'+q)(r'+r) = \text{constant} \quad . \tag{3.20}$$

Thus we can write

$$(q' + q)_x = (q' - q)\sqrt{k^2 + (q' + q)(r' + r)}\ , \tag{3.21}$$

$$k = \text{const.}$$

This is one half of the BT eqs we are looking for. The other half can be conveniently chosen to be (3.15), while (3.16-18) can be shown case by case to be already contained in (3.21) and (3.15). This will be demonstrated in the next section. Also Eq. (3.10) is now totally taken care of by (3.21) for each of the three kinds of r in (2.7).

In summary, the BT for the nonlinear equation characterized by (2.1) and (2.7) is given by

$$(q' + q)_x = (q' - q)\sqrt{k^2 + (q' + q)(r' + r)}\ , \tag{3.22a}$$

$$b_t = -\lambda(B' + B) + a(B' - B) + b(A' + A)\,, \tag{3.22b}$$

with

$$a = \frac{i(q' + q)_x}{2(q' - q)}\ , \qquad b = \frac{1}{2}(q' + q)\ . \tag{3.22c}$$

Notice that the X-dependence of Eq. (3.22b) is actually superficial, because all terms depending on λ will completely cancel out from (3.22b), as can be directly checked out for each of the five cases (2.2) – (2.6). Thus k is the only free parameter in the BT (3.22).

IV. DISCUSSIONS OF THE BACKLUND TRANSFORMATIONS

In this section we will take the K-dV eq. as an example to explicitly show that Eqs. (3.16-18) are indeed redundant, and hence Eqs. (3.3) are indeed completely reduced to Eqs. (3.22). This fact is also true for each of the other four cases (2.3) – (2.6), as can be directly verified.

For the K-dV eq., (3.22) becomes

$$(u' + u)_x = (u' - u)\sqrt{k^2 \pm 2(u' + u)}\ , \tag{4.1a}$$

$$(u' + u)_t = \frac{-(u' + u)_x (u' - u)_{xx}}{(u' - u)} \pm 3(u' + u)_x (u' + u)\ . \tag{4.1b}$$

Eqs. (3.16-18) are redundant, since (3.16) becomes now a trivial identity; (3.17) becomes

$$\frac{(u'_x)^2-(u_x)^2}{(u'-u)}-(u'+u)_{xx}\pm(u'-u)^2=0\quad ,$$

which is already contained in (4.1a); (3.18) can now be replaced by (3.19) plus (3.15) – (3.17), and (3.19) is again contained in (4.1 a).

To transform (4.1) into a more familiar form; we integrate (4.1) once over x and get

$$(w'+w)_x=(w'-w)^2-\frac{1}{4}k^2 \tag{4.2a}$$

$$(w'+w)_t=3(w'_x+w_x)^2+(w'_x-w_x)^2-2(w'-w)(w'-w)_{xx} \tag{4.2b}$$

where

$$w=\pm\frac{1}{2}\int^x u\,dx.$$

For the second examples, we turn to the BT for the sine-Gordon system. Eq. (3.22a) becomes

$$\frac{(q'+q)_x}{\sqrt{k^2-(q'+q)^2}}=(q'-q)\ ,$$

which can be integrated once over x to become

$$(u'+u)_x=2k\sin\left(\frac{u'-u}{2}\right) \tag{4.3a}$$

Eq. (3.22b) becomes $(u'+u)_{xt}=\sin u'+\sin u$, which combines with (4.3a) to give

$$(u'-u)_t=\frac{2}{k}\sin\left(\frac{u'+u}{2}\right)\quad . \tag{4.3b}$$

Eqs (4.3) are the familiar form of BT for the s-G eq..

In conclusion, we have obtained a universal formula of BT for an important class of nonlinear evolution equations. Our method essentially consists of making a special, simple ansatz on the λ-dependence of the Darboux matrix, Eq. (3.2). This ansatz probably has its roots in the Riemann-Hilbert problem of the integrable systems", and deserves some further study.

REFERENCES

1. See, for example, G. L. Lamb Jr., "Elements of Soliton Theory" (Wiley, New York 1980); M. J. Ablowitz and H. Segur, "Solitons and the Inverse Scattering Transform" (SIAM, Philadelphia 1981); F. Calogero and A. Degasperis, "Spectral Transform and Solitons I" (North-Holland, Amsterdam 1982); A. C. Newell, "Solitons in Mathematics

and Physics" (SIAM, Philadelphia 1985).

2. H. H. Chen, in "Backlund Transformations", R. M. Miura ed., Lecture Notes in Mathematics **5** 15 (Springer-Verlag, New York 1976).
3. K. Konno and M. Wadati, Prog. Theoret. Phys. 53, 1652 (1975).
4. D. Levi, 0. Ragnisco and A. Sym, Lett. Al Nuovo Cimento 33,401, (1982).
5. L.-L. Chau, in "Proc. Workshop on Vertex Operators in Mathematics & Physics (Berkeley, Nov. 1983)", eds. J. Lepowsky, S. Mandelstam and I. M. Singer (Springer-Verlag, New York, 1984).
6. V. E. Zakharov and A.V. Mikhailov, Sov. Phys. JETP 74, 1953 (1978); V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. 13, 166 (1979).