

## Geodesic Correspondence Transformation and Interacting Fields

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It is shown that by a geometric transformation of a flat space to a curved space we may determine the interactions of fields. In this transformation the coordinates are kept unchanged with an assumption that there is always a geodesic correspondence between the two spaces. The idea is illustrated by treating the cases of the gravitational field, electromagnetic interaction and of the strong interaction.

### I. INTRODUCTION

**D**URING the past fifty years, many efforts have been made in searching for a way to geometrize all interacting fields. Several works have been reported in finding out principles<sup>1),2)</sup> which may give the key of determining the field equations and the equations of motion. In the paper of Pandres<sup>1)</sup>, he used multivalued coordinates transformation as a basis to find out forces of fields. However, we use here another kind of treatment; we keep the coordinate unchanged, and let affine coefficients  $\bar{I}$  of a flat space transform to a curved space  $I'$ , under an assumption that there is always a geodesics correspondence between two spaces. When the equation of motion is transformed from the flat space ( $\bar{I} \equiv 0$ ) to the curved space, terms involving non-vanishing Christoffel coefficient will appear, which represent the interaction of fields.

The present paper aims only to illustrate a preliminary consequence of our idea, with treating the cases of gravitational fields, electromagnetic interaction and strong interaction as examples. The derivation of the field equations for the electromagnetic and strong interaction is not attempted here, and is reserved for a future publication.

### II. MATHEMATICAL FORMULATION

#### (1) Gravitation field

Consider two spaces with fundamental tensors  $\bar{g}$  and  $g$ , where the bar on  $g$  denotes, as usual, the Cartesian flat space. Those two spaces have the same Cartesian coordinates ( $x^i$ ). Then, in the flat space  $\bar{g}$ , the Christoffel coefficients are zero, and the equations of geodesic are

1) D. Pandres, Jr. *J. Math Phys.* 3, 602 (1962).

2) T. Fulton, F. Rohrlich and L. Witten, *Revs. Modern Phys.* 34, 442 (1962).

$$\frac{d^2x^i}{ds^2} = 0.$$

Assuming that for every geodesic of  $\bar{g}$  there exists always a corresponding geodesic in the  $g$  space. Then, the following relations hold<sup>3)</sup>:

$$\Gamma_{jk}^i = \bar{\Gamma}_{jk}^i + \delta_j^i \varphi_k + \delta_k^i \varphi_j,$$

where  $\varphi_i = \frac{1}{10} \frac{\partial \ln g}{\partial x^i}$  is a gradient vector. Since  $\bar{\Gamma}_{jk}^i \equiv 0$  for a flat space, we have

$$\Gamma_{jk}^i = \delta_j^i \varphi_k + \delta_k^i \varphi_j$$

Now we investigate the structure of this space, and prove that this space is a constant curvature space. From the well-known identity

$$\frac{\partial g_{ij}}{\partial x^k} = g_{ih} \Gamma_{jk}^h + g_{hj} \Gamma_{ik}^h,$$

we have

$$\frac{\partial g_{ij}}{\partial x^k} = 2g_{ij}\varphi_k + g_{ik}\varphi_j + g_{jk}\varphi_i, \tag{1}$$

and

$$\frac{\partial g_{ij}}{\partial x^l} = 2g_{ij}\varphi_l + g_{il}\varphi_j + g_{jl}\varphi_i. \tag{2}$$

Differentiating (1) with respect to  $x^l$ , and differentiating (2) with respect to  $x^k$ , and further from the commutative condition

$$\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 g_{ij}}{\partial x^l \partial x^k}, \tag{3}$$

we have

$$g_{ik}\varphi_{jl} + g_{kj}\varphi_{il} - g_{il}\varphi_{jk} - g_{lj}\varphi_{ik} = 0, \tag{4}$$

where

$$\varphi_{jl} \equiv \varphi_{,jl} - \varphi_{,j}\varphi_{,l}.$$

Multiplying (4) by  $g^{ik}$ , and then contracting, we have

$$4\varphi_{jl} + \varphi_{jl} - \varphi_{jl} - g_{jl}(g^{ik}\varphi_{ik}) = 0, \tag{5}$$

*i.e.*

$$\varphi_{jl} = \frac{1}{4}(g^{ik}\varphi_{ik})g_{jl} \tag{6}$$

From the definition of Riemann curvature tensor

$$R_{ijk}^h \equiv \Gamma_{ik}^h{}_{,j} - \Gamma_{ij}^h{}_{,k} + \Gamma_{ik}^m \Gamma_{jm}^h - \Gamma_{ij}^m \Gamma_{km}^h,$$

we have

$$R_{ijk}^h = \delta_k^h \varphi_{,ij} - \delta_j^h \varphi_{,ik} + \delta_j^h \varphi_{,i} \varphi_k - \delta_k^h \varphi_{,i} \varphi_j \tag{7}$$

3) L. P. Eisenhart, *Riemannian Geometry*. (Princeton University Press, Princeton, N.J. 1949) P. 132.

Multiplying by  $g_{hl}$ , then it becomes

$$R_{lijk} = g_{kl}\varphi_{ij} - g_{jl}\varphi_{ik}. \quad (8)$$

Substituting (6) into (8), we have

$$R_{lijk} = \rho(g_{kl}g_{ij} - g_{lj}g_{ik}), \quad (9)$$

or

$$R_{ij} = 3\rho g_{ij},$$

where  $R_{ij}$  are Ricci tensors and the curvature  $\rho = \frac{1}{4}g^{ik}\varphi_{ik}$  is obviously constant. This is the gravitational field equation.

Treating  $dx^i/ds$  as the field variables, we have, from the parallel displacement  $(dx^i/ds)_{;k} = 0$ ,

$$\frac{d^2x^i}{ds^2} + 2\varphi_j \frac{dx^i}{ds} \frac{dx^j}{ds} = 0, \quad (10)$$

which are the equations of motion of a particle in the gravitational field. The  $\varphi_i$ 's should be solved from the field equation Eq. (9)

## (2) Electromagnetic field

By an analogy to the gravitational field, we may obtain the field equations from the corresponding transformation, using spinor analysis instead of tensor calculus. The fundamental relations are

$$\gamma_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \gamma_\mu = 2g_{\mu\nu}I,$$

where  $\gamma$ 's are Dirac matrices.

Affine coefficient  $\Gamma_i$  are defined as<sup>4),5)</sup>

$$\frac{\partial \gamma_\mu}{\partial x^i} = \Gamma_i \gamma_\mu + \gamma_\mu \Gamma_i.$$

Consider there is another space with

$$\tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu = 2g_{\mu\nu}I,$$

and suppose that in the flat space

$$\tilde{\Gamma} \equiv 0.$$

Then by transforming it into the curved space we have

$$\Gamma_i = \tilde{\Gamma}_i + ieA_i I + i\rho\gamma_i,$$

where  $\rho$  is a constant and  $A_i$  the vector potential.

If  $\gamma_\mu \psi$  is treated as the field variables, then for parallel displacement  $(\gamma_\mu \psi)_{;i} \equiv 0$ , we obtain

$$\gamma_\mu \frac{\partial \psi}{\partial x^\mu} - ieA_\mu \gamma_\mu \psi - i\rho g_{\mu\mu} \psi = 0,$$

which is the familiar Dirac equation in the electromagnetic field.

4) W. L. Bade and Herbert Jehle, Revs. Modern Phys. 25, 714 (1953).

5) D. R. Brill and J. A. Wheeler, Revs. Modern Phys. 29, 465 (1957).

### (3) Strong interacting fields

Theoretical studies have suggested that strong interactions are basically vector in nature.<sup>6)</sup> In the recent investigation, Gell-Mann<sup>7)</sup> and Néeman<sup>8)</sup> have proposed the octet model to classify all particles and resonances. The vector mesons  $\rho$ ,  $\omega$  and  $K^*$ , may be put in a unitary octet, denoted by  $B_\mu$ . Since it has been shown further by Néeman that the theory of the strong fields can be developed in the same manner as that of an electromagnetic interaction, we can transform, as usual the flat space  $\bar{I}_i=0$  into the curved space  $I'$ , and have

$$I'_\mu = iGB_\mu + ik\gamma_\mu$$

Let  $\psi$  be the Fermion octet, i.e.  $\psi(p, n, \Xi^0, \Xi^-, \Sigma^+, \Sigma^-, \Sigma^0, \Lambda)$  and  $\gamma_\mu\psi$  the field variables. Then for parallel displacement, we have

$$\gamma_\mu \frac{\partial \psi}{\partial x^\mu} - iGB_\mu \gamma_\mu \psi - ikg_{\mu\mu} \psi = 0$$

which represents the interactions of strong field. It is expected that the present idea may cover the case of weak interaction.

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6) J. J. Sakurai, *Annals of Physics (New York)* 11, 1 (1961).

7) M. Gell-Mann, *Phys. Rev.* 125, 1067 (1962).

8) Y. Neeman, *Nucl. Phys.* 26, 222 (1961).