

On the Altarelli-Parisi Integro-Differential Equations*

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The mechanisms involved in the Altarelli-Parisi integro-differential equations are clearly exhibited. New expressions for the Altarelli-Parisi transition probabilities are obtained. All the logarithmic exponents are then reproduced naturally without any regularization. The number of valence quarks and the proton's momentum are automatically conserved. The conditions for probability conservation are also discussed in detail, which restrict the choice of regularization of Altarelli and Parisi.

1. INTRODUCTION

ALTARELLI and Parisi⁽¹⁾ have proposed a very simple way to formulate the Q^2 dependence of quark and gluon number densities inside the proton within the framework of quantum chromodynamics. In their formalism, they first proposed a transition probability density from the parton A to parton B , then wrote down the Altarelli-Parisi integro-differential equations. In determining the variation part of the transition probability, which is denoted by $(\alpha_s(t)/2\pi)P'_{BA}(x)$ in our notation with x equal to the momentum ratio of the final parton B to the initial parton A , they regularize the $1/1-z$, artificially, add $\delta(1-x)$ term in $P'_{qq}(x)$ and $P'_{gg}(x)$, and use the constraints imposed by the conservation of the valence quark and the conservation of the proton momentum. They show that the moment of their P'_{BA} are just the logarithmic exponents (anomalous dimensions) derived by Georgi and Politzer⁽²⁾ and by Gross and Wilczek⁽³⁾. But we see that their way of determining P'_{BA} is unnatural since (1) the variation mechanism is not clearly exhibited. (2) they regularize $1/1-z$ but not $1/z$. (3) $\delta(1-z)$ term is added by hand. (4) constraints are not automatically satisfied.

In the present paper, we demonstrate the variation mechanism clearly and thus obtain new expressions for P'_{qq} and P'_{gg} in a natural way without any regularization. The above mentioned constraints are then automatically satisfied and all anomalous dimensions are reproduced. We also discuss the conditions for probability conservation in detail and find a new constraint on the choice of regularization of Altarelli and Parisi. We also derive new expressions for $\Delta P'_{qq}$ and $\Delta P'_{gg}$ which give the Q^2 evolution of the parton polarization density which is defined as the difference between the parton number density with the positive helicity and that with the negative helicity.

In section 2, we present our formalism for both unpolarized and polarized partons. In section 3, we derive the conditions for the probability conservation. In the last section, we summarize and conclude our work.

2. FORMALISM AND THE LOGARITHMIC EXPONENTS

Following Altarelli and Parisi, we consider that the number of quarks and gluons as seen by the current changes by the following mechanisms: (1) A quark originally at higher momentum

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(1) G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298.

(2) H. Georgi and H. D. Politzer, Phys. Rev. D9 (1974) 416.

(3) D. J. Gross and F. Wilczek, Phys. Rev. D9 (1974) 980.

may lose momentum by radiating a gluon, (2) A gluon inside the proton may produce a quark-antiquark pair, and (3) A gluon may split into two gluons; We denote the transition probability density per unit t at order $a(t)$ of a proton A into a proton B with fraction \mathbf{Z} of the parent momentum by emitting a parton c by $P_{BA}(\mathbf{Z}) \alpha_s(t)/2\pi$ ($t = \ln Q^2/Q_0^2$, $-Q^2$ is the four momentum square of the virtual photon, and Q_0^2 the suitable normalization point). We note that in our definition $P_{BA}(\mathbf{Z})$ is always positive definite even if A and B represent the same type of parton such as P_{qq} and P_{GG} . According to the above mechanisms the rate of change of the number density of quark with fraction \mathbf{x} of the proton momentum with respect to t consists of three terms: (1) quarks with momentum \mathbf{x} are produced by quarks with higher momentum, which radiates a gluon. (2) quarks with momentum \mathbf{x} are produced by quark-antiquark creation of a gluon. (3) quarks originally with momentum \mathbf{x} change into a quark with less momentum by radiating a gluon. Therefore the master equation is given accordingly as follows

$$\frac{dq^i(\mathbf{x}, t)}{dt} = \frac{\alpha(t)}{2\pi} \left\{ \int_{\mathbf{x}}^1 \frac{dy}{y} \left[q^i(y, t) P_{qq} \left(\frac{\mathbf{x}}{y} \right) + G(y, t) P_{qG} \left(\frac{\mathbf{x}}{y} \right) \right] - q^i(\mathbf{x}, t) \int_0^1 dz P_{q,}(Z) \right\}, \quad (1)$$

where q^i and G are the number densities of quark and gluon respectively and quarks are assumed to be massless. The last term which is new, represents the removal of quarks originally with momentum \mathbf{x} . The equation (1) may be rewritten as

$$\frac{dq^i(\mathbf{x}, t)}{dt} = \frac{\alpha(t)}{2\pi} \int_{\mathbf{x}}^1 \frac{dy}{y} \left\{ q^i(y, t) \left[P_{qq} \left(\frac{\mathbf{x}}{y} \right) - \delta \left(1 - \frac{\mathbf{x}}{y} \right) \int_0^1 dz P_{qq}(Z) \right] + G(y, t) P_{qG} \left(\frac{\mathbf{x}}{y} \right) \right\}. \quad (2)$$

The rate of change of the non-singlet quark density defined by $q'_{NS}(\mathbf{x}, t) = q^i(\mathbf{x}, t) - q^i(\mathbf{x}, t)$ is then given by

$$\begin{aligned} \frac{dq'_{NS}(\mathbf{x}, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_{\mathbf{x}}^1 \frac{dy}{y} q'_{NS}(y, t) P_{NS} \left(\frac{\mathbf{x}}{y} \right), \\ P_{NS}(Z) &= P_{qq}(Z) - \delta(1-Z) \int_0^1 dy P_{qq}(y) \equiv P'_{qq}(Z) \end{aligned} \quad (3)$$

from the above expression (3), we immediately obtain

$$\int_0^1 dz P_{NS}(Z) = 0.$$

Therefore the number of valence quarks is automatically conserved.

Similarly, the rate of change gluon number density with respect to t consists of four terms: (1) gluons with momentum \mathbf{x} are radiated by quarks. (2) gluons with momentum \mathbf{x} are produced by gluons. (3) gluons originally with momentum \mathbf{x} split into two gluons and (4) gluons originally with momentum \mathbf{x} split into quark-antiquark pair. Therefore, the master equation for gluon number density is given accordingly as follows

$$\begin{aligned} \frac{dG(\mathbf{x}, t)}{dt} &= \frac{\alpha(t)}{2\pi} \left\{ \int_{\mathbf{x}}^1 \frac{dy}{y} \left[\sum_{i=1}^{2f} q^i(y, t) P_{Gq} \left(\frac{\mathbf{x}}{y} \right) + G(y, t) P_{GG} \left(\frac{\mathbf{x}}{y} \right) \right] \right. \\ &\quad \left. - G(\mathbf{x}, t) \frac{1}{2} \left[\int_0^1 dz P_{GG}(Z) + 2f \int_0^1 dz P_{qG}(Z) \right] \right\}, \end{aligned} \quad (4)$$

where f is the number of quark flavors. The last two terms which are new represent the removal of gluons originally with momentum \mathbf{x} , and the factor $1/2$ comes out because

$$\frac{\alpha(t)}{2\pi} \left[\int_0^{1/2} dz P_{GG}(Z) + f \int_0^1 dz P_{qG}(Z) \right]$$

represents the total transition probability density per unit t for a gluon splits into two gluons or quark-antiquark pair. The equation (4) may be rewritten as

$$\begin{aligned} \frac{dG(\mathbf{x}, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_{\mathbf{x}}^1 \frac{dy}{y} \left[\sum_{i=1}^{2f} q^i(y, t) P_{Gq} \left(\frac{\mathbf{x}}{y} \right) + G(y, t) P'_{GG} \left(\frac{\mathbf{x}}{y} \right) \right], \\ P'_{GG}(Z) &= P_{GG}(Z) - \delta(1-Z) \left[\frac{1}{2} \int_0^1 dy P_{GG}(y) + f \int_0^1 dy P_{qG}(y) \right]. \end{aligned} \quad (5)$$

We note that $\alpha(t)P'_{\sigma\sigma}(Z)/2\pi$ represents the contribution to the variation of the gluon number density from the mechanisms undergone by gluons at t , and the expression in (5) clearly exhibits these mechanisms. Momentum conservation in the vertices implies

$$\begin{aligned} P_{qq}(Z) &= P_{q\bar{q}}(1-Z), \\ P_{q\sigma}(Z) &= P_{\sigma\bar{q}}(1-Z), \\ P_{\sigma\sigma}(Z) &= P_{\sigma\bar{\sigma}}(1-Z), \end{aligned} \quad (6)$$

which are valid for all Z including $Z=1$. The equation (6) then implies

$$\begin{aligned} \int_0^1 dz Z P_{q\bar{q}}(Z) &= \int_0^1 dz (1-Z) P_{qq}(Z), \\ \int_0^1 dz Z P_{\sigma\bar{q}}(Z) &= \frac{1}{2} \int_0^1 dz P_{q\sigma}(Z), \\ \int_0^1 dz Z P_{\sigma\bar{\sigma}}(Z) &= \frac{1}{2} \int_0^1 dz P_{\sigma\sigma}(Z), \end{aligned} \quad (7)$$

and equations (2), (4) and (7) immediately give

$$\frac{d}{dt} \int_0^1 dx x \left[\sum_{i=1}^{2f} q^i(x, t) + G(x, t) \right] = 0. \quad (8)$$

Therefore, the proton's momentum is automatically conserved.

We define the logarithmic exponents as usual by

$$A_n \equiv \int_0^1 dz Z^{n-1} P(Z). \quad (9)$$

Then the expression (3) immediately gives

$$A_n^{\sigma\sigma} \equiv \int_0^1 dz Z^{n-1} P_{\sigma\sigma}(Z) = A_n^{\sigma\sigma} - A_1^{\sigma\sigma}. \quad (10)$$

Since

$$A_n^{\sigma\sigma} = \sum_{k=0}^{n-1} (-)^k C_k^{n-1} A_{k+1}^{\sigma\sigma},$$

as implied by the equation (6), we have

$$A_n^{\sigma\sigma} = \sum_{k=0}^{n-1} (-)^k C_k^{n-1} A_{k+1}^{\sigma\sigma}. \quad (11)$$

The $A_1^{\sigma\sigma}$ term disappears automatically in the left hand side of the equation (11) because $A_1^{\sigma\sigma} = A_1^{\sigma\sigma}$ as implied by the equation (6). The expression (5) gives immediately the following expression

$$A_1^{\sigma\sigma} \equiv \int_0^1 dz Z^{n-1} P'_{\sigma\sigma}(Z) = A_n^{\sigma\sigma} - \frac{1}{2} A_1^{\sigma\sigma} - f A_1^{\sigma\sigma}. \quad (12)$$

Altarelli and Parisi have derived the following expressions

$$\begin{aligned} P_{q\bar{q}}(Z) &= C_2(R) \frac{1+(1-Z)^2}{Z}, \\ P_{q\sigma}(Z) &= C_2(R) \frac{1+Z^2}{1-Z}, \\ P_{\sigma\bar{q}}(Z) &= \frac{1}{2} [Z^2 + (1-Z)^2], \\ P_{\sigma\bar{\sigma}}(Z) &= 2C_2(G) \left[\frac{1-Z}{Z} + \frac{Z}{1-Z} + Z(1-Z) \right], \end{aligned} \quad (13)$$

where $C_2(R) = (N^2 - 1)/2N$ with N as the number of colors. We note that the expression (13) is now valid for all Z including $Z=1$, where simple poles exist for $P_{q\bar{q}}$ and $P_{\sigma\bar{q}}$. The expression (13) is then used to calculate the logarithmic exponents. We then have

$$\begin{aligned}
A_n^{q^q} &= C_2(R) \left[2 - \frac{1}{n(n+1)} + 2 \sum_{i=2}^n \frac{1}{i} + 2 \lim_{Z \rightarrow 0} \ln Z \right], \quad n \geq 1 \\
A_n^{G^G} &= 2C_2(G) \left[-1 + \frac{1}{n(n-1)} + \frac{1}{(n+1)(n+2)} - \sum_{i=2}^n \frac{1}{i} - \lim_{Z \rightarrow 0} \ln Z \right], \quad n > 1 \\
A_1^{G^G} &= -2C_2(G) \left[\frac{11}{6} + 2 \lim_{Z \rightarrow 0} \ln Z \right],
\end{aligned} \tag{14}$$

where $C_2(G) = N$ is the number of color. The expressions (10) and (12) then give

$$\begin{aligned}
A_n^{NS} &= C_2(R) \left[-\frac{1}{2} + \frac{1}{n(n+1)} - 2 \sum_{i=2}^n \frac{1}{i} \right], \quad n \geq 1 \\
A_n^{G^G} &= C_2(G) \left[-\frac{1}{6} + \frac{2}{n(n-1)} + \frac{2}{(n+1)(n+2)} - 2 \sum_{i=2}^n \frac{1}{i} - \frac{2T(R)}{3C_2(G)} \right], \quad n > 1
\end{aligned} \tag{15}$$

where $2T(R) = f$ is the number of flavors. We see that the results are identical to the results of refs. (2,3). As far as $A_n^{G^G}$, there are no difference in calculation from that used by Altarelli and Parisi.

For singlet quark density, defined by the sum of all quark and antiquark densities, the equation is

$$\frac{dq^i(x, t)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[q^i(y, t) P_{NS}\left(\frac{x}{y}\right) + 2fG(y, t) P_{qG}\left(\frac{x}{y}\right) \right], \tag{16}$$

as implied by the equation (2). Then from the expressions (5) and (16), we have, as usual

$$\frac{d}{dt} \begin{pmatrix} M_n^+ \\ M_n^G \end{pmatrix} = \frac{\alpha(t)}{2\pi} \begin{pmatrix} A_n^{NS} & 2fA_n^{G^G} \\ A_n^{G^G} & A_n^{G^G} \end{pmatrix} \begin{pmatrix} M_n^+ \\ M_n^G \end{pmatrix}, \tag{17}$$

where the moment of parton densities are defined by

$$M_n^+ \equiv \int_0^1 dz Z^{n-1} q^+(x, t),$$

etc.

For spin-dependent case, we denote, by $q^\lambda(x, t)$ the parton densities with helicity $\lambda = \pm$ in a proton of positive helicity. The parity conservation implies that

$$P_{A+B\pm}(Z) = P_{A-B\pm}(Z) \tag{18}$$

for any parton A and B. This in turn implies

$$\begin{aligned}
P_{A+B+} + P_{A-B+} &= P_{A-B-} + P_{A+B-} \equiv P_{AB}, \\
P_{A+B+} - P_{A-B+} &= -(P_{A+B-} - P_{A-B-}) \equiv \Delta P_{AB},
\end{aligned} \tag{19}$$

where P_{AB} is the same one as that used before. Following the arguments presented in this note, we write down the following equations

$$\begin{aligned}
\frac{dq_\lambda^i(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \left\{ \int_x^1 \frac{dy}{y} \left[q_\lambda^i(y, t) P_{q_\lambda q_\lambda}\left(\frac{x}{y}\right) + q_\lambda^i(x, t) P_{q_\lambda q_\lambda}\left(\frac{x}{y}\right) + G_\pm(y, t) P_{q_\lambda G_\pm}\left(\frac{x}{y}\right) \right. \right. \\
&\quad \left. \left. + G_\mp(y, t) P_{q_\lambda G_\mp}\left(\frac{x}{y}\right) \right] - q_\lambda^i(x, t) \int_0^1 dz P_{qG}(Z) \right\}, \\
\frac{dG_\lambda(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \left\{ \int_x^1 \frac{dy}{y} \left[q_\lambda^i(y, t) P_{G_\lambda q_\lambda}\left(\frac{x}{y}\right) + q_\lambda^i(y, t) P_{G_\lambda q_\lambda}\left(\frac{x}{y}\right) + G_\pm(y, t) P_{G_\lambda G_\pm}\left(\frac{x}{y}\right) \right. \right. \\
&\quad \left. \left. - G_\mp(y, t) P_{G_\lambda G_\mp}\left(\frac{x}{y}\right) \right] - G_\lambda(x, t) \left[\frac{1}{2} \int_0^1 dz P_{GG}(Z) + f \int_0^1 dz P_{qG}(Z) \right] \right\},
\end{aligned} \tag{20}$$

where $\lambda = \pm$ is the helicity. Parity conservation implies that the sums $q^i(x, t) = q_+^i(x, t) + q_-^i(x, t)$ and $G(x, t) = G_+(x, t) + G_-(x, t)$ and the differences $\Delta q^i(x, t) = q_+^i(x, t) - q_-^i(x, t)$ and $\Delta G = G_+(x, t) - G_-(x, t)$ evolve separately. The evolution of the sums has been discussed before. The evolution of the differences is easily derived from the expressions (20) as follows

$$\begin{aligned}\frac{d\Delta q^i(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[\Delta q^i(y, t) \Delta P_{NS} \left(\frac{x}{y} \right) + \Delta G(y, t) \Delta P_{q\sigma} \left(\frac{x}{y} \right) \right], \\ \frac{d\Delta G(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[\Delta q^i(y, t) \Delta P_{\sigma q} \left(\frac{x}{y} \right) + \Delta G(y, t) \Delta P'_{\sigma\sigma} \left(\frac{x}{y} \right) \right],\end{aligned}\quad (21)$$

where

$$\begin{aligned}\Delta P_{NS}(Z) &= \Delta P_{qq}(Z) - \delta(1-Z) \int_0^1 dy P_{qq}(y), \\ \Delta P'_{\sigma\sigma}(Z) &= \Delta P_{\sigma\sigma}(Z) - \delta(1-Z) \int_0^1 dy \left[\frac{1}{2} P_{\sigma\sigma}(y) + f P_{q\sigma}(y) \right].\end{aligned}\quad (22)$$

The expression (21) then gives

$$\frac{d\Delta q'_{NS}(x, t)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \Delta q'_{NS}(y, t) \Delta P_{NS} \left(\frac{x}{y} \right).\quad (23)$$

Since, when quark masses are neglected, P_{q-q+} vanishes owing to helicity conservation, therefore P_{qq} is equal to ΔP_{qq} which then implies

$$\Delta P_{NS} = P_{NS}\quad (24)$$

This immediately gives

$$\tilde{A}_n^{NS} \equiv \int_0^1 dz Z^{n-1} \Delta P_{NS} = A_n^{NS}.\quad (25)$$

It has been derived by Altarelli and Parisi that

$$\begin{aligned}P_{\sigma+\sigma+}(Z) &= C_2(G) (1+Z^4) \left(\frac{1}{Z} + \frac{1}{1-Z} \right), \\ P_{\sigma-\sigma+}(Z) &= C_2(G) \frac{(1-Z)^3}{Z}.\end{aligned}\quad (26)$$

This then gives

$$\Delta P_{\sigma\sigma}(Z) = 2C_2(G) \left[\frac{1}{1-Z} + 1 - 2Z \right].\quad (27)$$

which in turn implies

$$\tilde{A}_n^{\sigma\sigma} \equiv \int_0^1 dz Z^{n-1} \Delta P_{\sigma\sigma} = 2C_2(G) \left[\frac{1}{n} - \frac{2}{n+1} - \sum_{i=1}^{n-1} \frac{1}{i} - \lim_{Z \rightarrow 0} \ln Z \right].\quad (28)$$

From the expressions (22), (14) and (28), we obtain

$$\begin{aligned}\tilde{A}_n^{\sigma\sigma} &\equiv \int_0^1 dz Z^{n-1} \Delta P'_{\sigma\sigma} = \tilde{A}_n^{\sigma\sigma} - \frac{1}{2} A_1^{\sigma\sigma} - f A_1^{q\sigma} \\ &= C_2(G) \left[\frac{2}{n} - \frac{4}{n+1} - 2 \sum_{i=1}^{n-1} \frac{1}{i} + \frac{11}{6} - \frac{2T(R)}{3C_2(G)} \right].\end{aligned}\quad (29)$$

The expressions (25) and (29) are just the results obtained by refs.^(4,5). Other logarithmic exponents can be calculated in the same way as done by Altarelli and Parisi. Finally, we note that the expression (21) immediately gives

$$\frac{d}{dt} \begin{pmatrix} \tilde{M}_n^i \\ \tilde{M}_n^\sigma \end{pmatrix} = \frac{\alpha(t)}{2\pi} \begin{pmatrix} A_n^{NS} & 2f\tilde{A}_n^{\sigma\sigma} \\ \tilde{A}_n^{\sigma q} & \tilde{A}_n^{\sigma\sigma} \end{pmatrix} \begin{pmatrix} \tilde{M}_n^i \\ \tilde{M}_n^\sigma \end{pmatrix},\quad (30)$$

where

$$\begin{aligned}\tilde{M}_n^i &\equiv \int_0^1 dz Z^{n-1} \Delta q^i(Z, t), \\ \tilde{M}_n^\sigma &\equiv \int_0^1 dz Z^{n-1} \Delta G(Z, t).\end{aligned}$$

(4) M.A. Ahmed and G.C. Ross, Phys. Letters 56B (1975) 385; Nucl. Phys. B111 (1976) 441.

(5) K. Sasaki Kyoto preprint KUNS 318.

3. CONSERVATION OF PROBABILITY

The transition probability density up to the first order in $\alpha_s(t)$ from the parton i at scale T_0 to parton a at scale T with momentum fraction x of the initial parton i is

$$\bar{D}_{a,i}(x, T) = \delta_{ai} \delta(1-x) + \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} P_{ai}'(x), \quad (31)$$

which is diagrammatically represented in Fig. 1 (A). If the initial parton is the quark q_i . Then the total transition probability is

$$\sum_{a \neq G} \int_0^1 dx \bar{D}_{a,q_i}(x, T) = 1, \quad (32)$$

where gluon is excluded from the sum to avoid double counting. If the initial parton is the gluon G . Then the total probability is

$$\int_{1/2}^1 dx \bar{D}_{G,G}(x, T) + f \int_0^1 dx \bar{D}_{q,G}(x, T) = 1, \quad (33)$$

where f is the number of quark flavours. In the expression (33), we sum over the final gluon and quarks but not antiquarks to avoid double counting. The integration limit $(1/2, 1)$ of the first term in the expression (33) is due to the fact that the P_{GG} term in \bar{D}_{GG} contributes $\int_{1/2}^1 dx P_{GG}(x)$ to the total probability and for the remaining $\delta(1-x)$ term, the integration limits $(0,1)$ and $(1/2, 1)$ give the same result.

The expression (32) then implies

$$\int_0^1 dx P_{q_i}'(x) = 0, \quad (34)$$

and the expression (33) implies

$$\int_{1/2}^1 dx P_{GG}'(x) + f \int_0^1 dx P_{q,G}'(x) = 0. \quad (35)$$

Therefore the expression (34) and (35) are the necessary and sufficient conditions for the probability to be conserved up to the first order in $\alpha_s(t)$.

In order to discuss the conservation of probability to all order in $\alpha_s(t)$, we denote $D_{a,i}(x, T)$ as the transition probability density to all order in $\alpha_s(t)$, from the parton i at scale T_0 to the parton a at scale T with momentum fraction x of the initial parton i . $D_{a,i}(x, T)$ is diagrammatically

$$\begin{aligned} \text{(A)} \quad & \underline{i \quad a} \quad + \quad \underline{i \quad \text{loop} \quad a} \\ \text{(B)} \quad & \underline{i \quad a} \quad \equiv \quad \underline{i \quad a} \quad + \quad \underline{i \quad \text{loop} \quad a} \quad + \quad \underline{i \quad \text{loop} \quad \text{loop} \quad a} \quad + \dots \\ & = \quad \underline{i \quad a} \quad + \quad \underline{i \quad \text{loop} \quad a} \\ & = \quad \underline{i \quad a} \quad + \quad \underline{i \quad \text{loop} \quad a} \\ \text{(C)} \quad & \underline{i \quad j \quad a} \quad + \quad \underline{i \quad j \quad \text{loop} \quad a} \end{aligned}$$

Fig. 1. Diagrammatical representations of (A) $\bar{D}_{a,i}(x, T)$, (B) $D_{a,i}(x, T)$, and (C) $D_{a,I,i}(y, z, T)$.

represented in the Fig. 1 (B) as in the reference⁽⁶⁾, in which the authors use the $P'_{a,i}$ determined by Altarelli and Parisi. We use the last equality in the Fig. 1 (B), which is suitable for the discussion on the conservation of probability. The expression for $D_{a,i}(x, T)$ is then

$$D_{a,i}(x, T) = \delta_{ai} \delta(1-x) + \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} \sum_j \int_x^1 \frac{dy}{y} D_{j,i}(y, t) P'_{a,j}\left(\frac{x}{y}\right), \quad (36)$$

where the summation is over all partons. In order to write down the total transition probability, we define $D_{a,j,i}(y, z, T)$ in the Fig. 1 (C) as the transition probability density for $i(T_0) \rightarrow j \rightarrow a(T)$ in which the intermediate parton j has a momentum fraction y of the initial parton i and the final parton a has a momentum fraction z of the intermediate parton j . The expression is then

$$D_{a,j,i}(y, z, T) = \delta_{ji} \delta_{af} \delta(1-y) \delta(1-z) + \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} D_{j,i}(y, t) P'_{a,j}(z). \quad (37)$$

The expressions (36) and (37) then imply

$$D_{a,i}(x, T) = \sum_j \int_0^1 dy \int_0^1 dz D_{a,j,i}(y, z, T) \delta(yz-x) dz. \quad (38)$$

The total transition probability from the parton i at scale T_0 to any parton at scale T is then

$$1 = \sum_{\substack{q \neq G \\ j \neq G}} \int_0^1 dy \int_0^1 dz D_{a,i}(y, z, T) \\ + \int_{1/2}^1 dz \int_0^1 dy D_{G,a,i}(y, z, T) + \sum_q \int_0^1 dz \int_0^1 dy D_{q,G,i}(y, z, T), \quad (39)$$

where \sum_q sums over quarks only i , e antiquarks and gluon are not included. This expression for the total probability is obtained because the final step of transition $j \rightarrow a(T)$ is a transition up to the first order, in $\alpha_s(a)$ which implies that (1) if $j \neq G$, then we sum only $a \neq G$ to avoid double counting. (2) if $j = G$, we use integration limits $(1/2, 1)$ to avoid double counting in $P_{G\sigma}$ term. (3) if $j = G$ and $a \neq G$, we sum only quarks (or only antiquarks) to avoid double counting. By using the expression (37), the expression (39) is reduced to the following form

$$\sum_{j \neq G} \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} \int_0^1 dy D_{j,i}(y, t) \int_0^1 dz P'_{q\sigma}(Z) \\ + \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} \int_0^1 dy D_{G,i}(y, t) \left[\int_{1/2}^1 dz P'_{G\sigma}(Z) + f \int_0^1 dz P_{q\sigma}(Z) \right] = 0 \quad (40)$$

which is satisfied if the expressions (34) and (35) are satisfied. Therefore we conclude that the necessary and sufficient conditions for the probability to be conserved to any order are given by the expression (34) and (35).

Our new expressions for $P'_{q\sigma}$ and $P'_{G\sigma}$ satisfy the constraints (34) and (35). The original Altarelli-Parisi expressions also satisfy these constraints if one uses the following regularization⁽⁷⁾

$$\int_x^1 dz Z \frac{H(x/z)}{(1-Z)_+} = H(x) \ln(1-x) + \int_x^1 \frac{dz}{1-Z} [ZH(x/z) - H(x)],$$

where $H(x)$ is any function regular at the endpoints.

4. SUMMARY AND CONCLUSIONS

In the present paper we demonstrate clearly the variation mechanisms involved in the Altarelli-Parisi integro-Differential equations, and thus obtain in a natural way new expressions for $P'_{q\sigma}$, $P'_{G\sigma}$, $\Delta P'_{q\sigma}$, and $\Delta P'_{G\sigma}$ which are given by the expressions (3), (5), and (22) respectively. Our new expressions have no regularization, and satisfy automatically the conservation of probability, the conservation of the valence quark and the conservation of the momentum, and reproduce all anomalous

(6) K. Konishi, A. Ukawa and G. Veneziano, Rutherford preprint RL-79-026. G. Veneziano, CERN preprint Ref. TH. 2691-CERN, 1979.

(7) Andrzej J. Buras, Rev. Mod. Phys 52 (1980) 199.

$$\begin{aligned}
\text{(A)} \quad \frac{i \text{ --- } a}{\times} &= \sum \frac{i\lambda \text{ --- } a\lambda'}{\times} \\
\text{(B)} \quad \frac{i \text{ --- } a}{\times} &= \sum \frac{i\lambda \text{ --- } a\lambda'}{\times} \\
\text{(C)} \quad \frac{i \text{ --- } a}{\times} &= \sum \frac{i\lambda \text{ --- } a\lambda'}{\times} \\
\text{(D)} \quad \frac{i\lambda \text{ --- } a\lambda'}{\times} &= \frac{i\lambda \text{ --- } a\lambda'}{\times} + \frac{i\lambda \text{ --- } a\lambda'}{\times} + \frac{i\lambda \text{ --- } a\lambda'}{\times} + \dots \\
&= \frac{i\lambda \text{ --- } a\lambda'}{\times} + \frac{i\lambda \text{ --- } a\lambda'}{\times}
\end{aligned}$$

Fig. 2. Helicity decompositions for (A) P_{ai} , (B) P'_{ai} , (C) $D_{a,i}$ (D) is the transition probability density for $i_\lambda(T_0) \rightarrow a_{\lambda'}(T)$.

dimensions, while the Altarelli-Parisi expressions involve unnatural regularization in $1/1-z$ which the distribution in z , and use the conservation of the valence quark and the conservation of the momentum as constraints. We also derive the necessary and sufficient conditions of probability conservation. One of them (the condition (35)) is new and restricts the choice of regularization of Altarelli and Parisi.

We note that our results may be summarized by the following expressions

$$P'_{a_\lambda, i_\lambda}(x) = P_{a_\lambda, i_\lambda}(x) - \delta_{\lambda'\lambda} \delta_{ai} \delta(1-x) \begin{cases} \int_0^1 dz P_{qq}(Z), & i=q_i \\ \int_{1/2}^1 dx P_{Gq}(x) + f \int_0^1 dx P_{qG}(x), & i=G \end{cases} \quad (41)$$

where Γ 's are helicities and the P_{ai} , P'_{ai} , ΔP_{ai} and $\Delta P'_{ai}$ are given by

$$\begin{aligned}
P_{ai}(x) &= \sum_{\lambda'} P_{a_\lambda, i_\lambda}(x), \\
P'_{ai}(x) &= \sum_{\lambda'} P'_{a_\lambda, i_\lambda}(x), \\
\Delta P_{ai}(x) &= P_{a_+, i_+}(x) - P_{a_-, i_+}(x), \\
\Delta P'_{ai}(x) &= P'_{a_+, i_+}(x) - P'_{a_-, i_+}(x).
\end{aligned} \quad (42)$$

The first two equalities in expression (42) are represented in Fig. 2 (A) and (B) respectively. Similarly, we write the transition probability as

$$D_{a,i}(x, T) = \sum_{\lambda'} D_{a_\lambda, i_\lambda}(x, T), \quad (43)$$

with

$$D_{a_\lambda, i_\lambda}(x, T) = \delta_{\lambda'\lambda} \delta_{ai} \delta(1-x) + \int_{T_0}^T dt \frac{\alpha_s(t)}{2\pi} \sum_{j, \lambda'} \int_0^1 \frac{dy}{y} D_{j, \lambda', i_\lambda}(y, t) P'_{a_\lambda, j, \lambda'}\left(\frac{x}{y}\right). \quad (44)$$

which are represented in Fig. 2 (C) and (D) respectively.

The parton number density with definite helicity, denoted by $G_k^{i_\lambda}(x, T)$, is then given by

$$G_k^{i_\lambda}(x, T) = \sum_{i, \lambda'} \int_x^1 \frac{dy}{y} G_k^{i_\lambda'}\left(\frac{x}{y}, T_0\right) D_{a_\lambda, i_\lambda'}(y, T), \quad (45)$$

which implies

$$\frac{dG_k^{i_\lambda}(x, t)}{dt} = \frac{\alpha_s(t)}{2\pi} \sum_{j, \lambda'} \int_x^1 \frac{dz}{Z} G_k^{j, \lambda'}\left(\frac{x}{Z}, t\right) P'_{a_\lambda, j, \lambda'}(Z). \quad (46)$$

The expressions (41) to (46) then reproduce all results presented in the present paper.

Note added: After the finish of this paper, we have been called attention to the paper written by J. Wesiek and K. Zalewski (Nucl. Phys. B 161 (1979) 294), in which the same P'_{qq} and P'_{Gq} (in our notation) can be obtained.