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Dispersion Relation in a Bi-Maxwellian Magnetoplasma

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Plasma wave dispersion relations are considered for a thermally anisotropic collisionless plasma in a uniform magnetic field. Assuming the zero-order distribution function of each charge species in the plasma is bi-Maxwellian, the dispersion relations are obtained in terms of the plasma dispersion function and or the Bessel functions. Explicit expressions are given for cylcotron waves, hybrid mode, transverse mode and electrostatic mode.

I. INTRODUCTION

NATURALLY Occuring plasmas in space such as solar wind, magnetospheric plasma, and some laboratory produced plasma from recent experimental developments often show that they are thermally anisotropic in the presence of a magnetic field. Their velocity distributions are of Bi-Maxwellian or nearly Bi-Maxwellian. Although the general plasma wave dispersion relation is well known[1~3], very few detailed and well organized explicit expression in terms of the plasma dispersion function defined by Fried and Conte[4] can be found for a plasma of Bi-Maxwellian distribution in a magnetic field. This paper is to give plasma wave dispersion relations for a collisionless magnetoplasma of Bi-Maxwellian distribution in terms of the plasma function and or the Bessel functions. The resulting expressions can be used later for work on plasma waves and instabilities in a magnetoplasma of Bi-Maxwellian distribution and also waves in the solar wind.

II. GENERAL DISPERSION RELATIONS

Consider a collisionless magnetoplasma in which binary collisions have no significant effect on its oscillatory properties. The zero-order quantities are uniform in space and are time independent. The first-order quantities vary as $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Put the Cartesian coordinate system in such a way that the uniform magnetic feld $\mathbf{B}_0$ is in the direction of z axis and the wave propagation rector k is in xz plane making an angle θ with $\mathbf{B}_0$,

$$\mathbf{k}=k_{\perp}\hat{\mathbf{x}}+k_{\parallel}\hat{\mathbf{z}}=k\sin\theta\hat{\mathbf{x}}+k\cos\theta\hat{\mathbf{z}},$$

$$\mathbf{B}_0=B_0\hat{\mathbf{z}},$$

$$\mathbf{n}\equiv\frac{\mathbf{k}c}{\omega}=n_{\perp}\hat{\mathbf{x}}+n_{\parallel}\hat{\mathbf{z}}=n\sin\theta\hat{\mathbf{x}}+n\cos\theta\hat{\mathbf{z}},$$

where n is refractive index. The notations $\perp$ and $\parallel$ refer to perpendicular and parallel component

(1) T.H. Stix, ***The Theory of Plasma Waves,*** pp. 188-194, (McGraw-Hill, New York. 1962).
(2) W.P. Allis, S. J. Buchsbaum, and A. Bers, ***Waves in Anisotropic Plasmas,*** pp. 11-13, (The M.I.T. Press, Cambridge, 1963).
(3) A.I. Akhiezer, I.A. Akhiezer, R.V. Polovin, A.G. Sitenko, and K. N. Stepanov, ***Collective Oscillations in a Plasma, pp. 26-31,*** (The M. I.T. Press, Cambridge, 1967).
(4) B. D. Fried and S. D. Conte, ***The Plasma Dispersion Function***, (Academic Press, New York, 1961).

with respect to $\mathbf{B}_0$. The general plasma wave dispersion relation[2] is

$$\begin{vmatrix} K_{xx}-n_{\parallel}^2 & K_{xy} & K_{xz}+n_\perp n_\parallel \\ K_{yx} & K_{yy}-n^2 & K_{yz} \\ K_{zx}+n_\perp n_\parallel & K_{zy} & K_{zz}-n_\perp^2 \end{vmatrix} = o, \tag{1}$$

or

$$An^4+Gn^2+C=0, \tag{2}$$

where

A- $K_{xx}\sin^2$ e $+2K_{xz}\sin$ e cos e + $K_{zz}\cos^2\theta$,

$G=-K_{xx}K_{zz}+K_{xz}^2-(K_{xx}K_{yy}+K_{xy}^2)$ sin" $\theta+(K_{xy}K_{yz}-K_{xz}K_{yy})\sin 2\theta-(K_{yy}K_{zz}+K_{yz}^2)\cos^2\theta$,

$C=K_{zz}(K_{xx}K_{yy}+K_{xy}^2)+K_{xx}K_{yz}^2+2K_{xy}K_{xz}K_{yz}-K_{yy}K_{xz}^2$.

The dielectric tensor $K_{\alpha\beta}$ can be obtained from the zero-order velocity distribution function of each charge species in the plasma[3]:

$$K_{xx}=1+2\pi\sum_j\left(\frac{\Omega_j\pi_j}{k_\perp\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell^2\int_0^\infty J_\ell^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)dv_\perp\int_{-\infty}^{\infty}\frac{(\omega-k_\parallel v_\parallel)\frac{\partial f_{oj}}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}dv_\parallel,$$

$$K_{yy}=1+2\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\int_0^\infty J_\ell'^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)v_\perp^2dv_\perp\int_{-\infty}^{\infty}\frac{(\omega-k_\parallel v_\parallel)\frac{\partial f_{oj}}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}dv_\parallel,$$

$$K_{zz}=1-2\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\left[\frac{1}{2\pi}-\int_0^\infty dv_\perp\sum_{\ell=-\infty}^{\infty}J_\ell^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)\int_{-\infty}^{\infty}\frac{k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}+\ell\Omega_j\frac{\partial f_{oj}}{\partial v_\perp}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}v_\parallel^2dv_\parallel\right],$$

$$K_{xy}=-K_{yx}=2\pi i\sum_j\frac{\epsilon_j\Omega_j}{k_\perp}\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell\int_0^\infty J_\ell\left(\frac{k_\perp v_\perp}{\Omega_j}\right)J_\ell'\left(\frac{k_\perp v_\perp}{\Omega_j}\right)v_\perp dv_\perp\int_{-\infty}^{\infty}\frac{(\omega-k_\parallel v_\parallel)\frac{\partial f_{oj}}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}dv_\parallel,$$

$$K_{xz}=K_{zx}=2\pi\sum_j\frac{\Omega_j}{k_\perp}\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell\int_0^\infty J_\ell^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right)dv_\perp\int_{-\infty}^{\infty}\frac{(\omega-k_\parallel v_\parallel)\frac{\partial f_{oj}}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}v_\parallel dv_\parallel.$$

$$K_{yz}=-K_{zy}=-2\pi i\sum_j\epsilon_j\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\int_0^\infty J_\ell\left(\frac{k_\perp v_\perp}{\Omega_j}\right)J_\ell'\left(\frac{k_\perp v_\perp}{\Omega_j}\right)v_\perp dv_\perp\int_{-\infty}^{\infty}\frac{(\omega-k_\parallel v_\parallel)\frac{\partial f_{oj}}{\partial v_\perp}+k_\parallel v_\perp\frac{\partial f_{oj}}{\partial v_\parallel}}{\omega-k_\parallel v_\parallel-\ell\Omega_j}v_\parallel dv_\parallel.$$

(3)

where $J_\ell\left(\frac{k_\perp v_\perp}{\Omega_j}\right)$ is the Bessel function of order ℓ : $J_\ell'\left(\frac{k_\perp v_\perp}{\Omega_j}\right)$ is the derivative of the Bessel function with respect to its argument; $f_{oj}(v_\perp, v_\parallel)$ is the zero-order velocity distribution function of the "*j*" charge species in the plasma; ϵ_j is the sign of the "*j*" charge species, i. e. $\epsilon_j=+1$ fore positive "*j*" charge species and $\epsilon_j=-1$ for negative "*j*" charge species. The cyclotron frequency Ω_j and the plasma frequency π_j of the "*j*" charge species are defined respectively as

$$\Omega_j\equiv\left|\frac{q_jB_o}{m_jc}\right|, \qquad \pi_j\equiv\left(\frac{4\pi N_jq_j^2}{m_j}\right)^{1/2}.$$

In electrostatic approximation the electric wave vector is nearly parallel to the propagation vector k. The refractive index *n* is very large. The oscillations of the plasma is almost longitudinal. The dispersion relation for electrostatic mode is thus obtained from equation (2) by putting *A=O:*

$K_{xx}\sin^2$ e $+2K_{xz}\sin$ e cos e $+K_{zz}\cos^2$ e= 0, (4)

or

$$k_\perp^2K_{xx}+2k_\perp k_\parallel K_{xz}+k_\parallel^2K_{zz}=0. \tag{4'}$$

1. Parallel Propagation

For propagation parallel to the magnetic field $\mathbf{B}_0$, we have.

$$\theta=0,\quad k=k_{\parallel},\quad k_{\perp}=0,\quad n=n_{\parallel},\quad n_{\perp}=0.$$
$$K_{xz}=K_{yz}=K_{zx}=K_{zy}=0,\quad K_{xx}=K_{yy}.$$

Equation (I) becomes

$$\begin{vmatrix} K_{xx}-n^2 & K_{xy} & 0 \\ -K_{xy} & K_{xx}-n^2 & 0 \\ 0 & 0 & K_{zz} \end{vmatrix}=0,$$

or

$$K_{zz}[(n^2-K_{xx})^2+K_{xy}^2]=0.$$

The dispersion relation for electrostatic mode is

$$K_{zz}=0 \tag{5}$$

From (3) and (5). we have

$$1-2\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\left[\frac{1}{2\pi}-k\int_0^{\infty}v_{\perp}dv_{\perp}\int_{-\infty}^{\infty}\frac{v_{\parallel}^2\dfrac{\partial f_{oj}}{\partial v_{\parallel}}}{\omega-kv_{\parallel}}dv_{\parallel}\right]=0. \tag{5'}$$

The dispersion relation for electrostatic mode can also be obtained from equation (49. In this mode the electric wave field is parallel to the magnetic field $\mathbf{B}_0$. The other dispersion relation is for cyclotron waves:

$$(n^2-K_{xx})^2+K_{xy}^2=0,$$

or

$$n^2=K_{xx}\pm iK_{xy}.\qquad \Big\}\ \begin{matrix}\text{RH.}\\ \text{LH}\end{matrix} \tag{6}$$

From (3) and (6). we have

$$\left(\frac{kc}{\omega}\right)^2=1+\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\int_{-\infty}^{\infty}dv_{\parallel}\int_0^{\infty}\left[(\omega-kv_{\parallel})\frac{\partial f_{oj}}{\partial v_{\perp}}+kv_{\perp}\frac{\partial f_{oj}}{\partial v_{\parallel}}\right]\frac{v_{\perp}^2dv_{\perp}}{\omega-kv_{\parallel}\pm\epsilon_j\Omega_j}.\ \Big\}\ \begin{matrix}\text{RH}\\ \text{LH}\end{matrix} \tag{6'}$$

Upper sign is for the right-hand (RH) circular polarized cyclotron wave and lower sign is for the left-hand (LH) circular polarized cyclotron wave with respect to $\mathbf{B}_0$.

2. **Perpendicular Propagation**

For propagation perpendicular to the magnetic field $\mathbf{B}_0$, we have

$$\theta=\frac{\pi}{2},\quad k=k_{\perp},\quad k_{\parallel}=0,\quad n=n_{\perp},\quad n_{\parallel}=0,$$
$$K_{xz}=K_{zx}=K_{yz}=K_{zy}=0.$$

Equation (1) becomes

$$\begin{vmatrix} K_{xx} & K_{xy} & 0 \\ -K_{xy} & K_{yy}-n^2 & 0 \\ 0 & 0 & K_{zz}-n^2 \end{vmatrix}=0,$$

or

$$(K_{zz}-n^2)[K_{xx}(K_{yy}-n^2)+K_{xy}^2]=0.$$

The dispersion relation for transverse mode is

$$n^2 = K_{zz}. \tag{7}$$

From (3) and (7), we have

$$\left(\frac{kc}{\omega}\right)^2 = 1 - \sum_j \left(\frac{\pi_j}{\omega}\right)^2 + 2\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\int_0^\infty dv_\perp \int_{-\infty}^\infty dv_\parallel \sum_{\ell=-\infty}^{\infty} J_\ell^2\left(\frac{kv_\perp}{\Omega_j}\right)\frac{\ell\Omega_j v_\parallel^2 \frac{\partial f_{oj}}{\partial v_\perp}}{\omega-\ell\Omega_j}. \tag{7'}$$

The transverse mode sometimes_ called ordinary wave is linearly polarized with electric wave field parallel to $\mathbf{B}_0$. This mode has been discussed by Landau and Cuperman[5]. The other dispersion relation is for hybrid mode:

$$K_{xx}(K_{yy}-n^2)+K_{xy}^2=0,$$

or

$$n^2 = K_{yy} + \frac{K_{xy}^2}{K_{xx}}, \tag{8}$$

where

$$K_{xx} = 1+2\pi\sum_j\left(\frac{\Omega_j\pi_j}{k\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell^2\int_0^\infty J_\ell^2\left(\frac{kv_\perp}{\Omega_j}\right)dv_\perp\int_{-\infty}^\infty\frac{\omega}{\omega-\ell\Omega_j}\frac{\partial f_{oj}}{\partial v_\perp}dv_\parallel,$$

$$K_{yy} = 1+2\pi\sum_j\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\int_0^\infty J_\ell'^2\left(\frac{kv_\perp}{\Omega_j}\right)v_\perp^2 dv_\perp\int_{-\infty}^\infty\frac{\omega}{\omega-\ell\Omega_j}\frac{\partial f_{oj}}{\partial v_\perp}dv_\parallel,$$

$$K_{xy} = 2\pi i\sum_j\frac{\epsilon_j\Omega_j}{k}\left(\frac{\pi_j}{\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell\int_0^\infty J_\ell\left(\frac{kv_\perp}{\Omega_j}\right)J_\ell'\left(\frac{kv_\perp}{\Omega_j}\right)v_\perp dv_\perp\int_{-\infty}^\infty\frac{\omega}{\omega-\ell\Omega_j}\frac{\partial f_{oj}}{\partial v_\perp}dv_\parallel.$$

The hybrid mode shows elliptical polarization in a plane perpendicular to the magnetic field $\mathbf{B}_0$. It couples electrostatic mode and transverse electromagnetic. mode and is sometimes called extraordinary wave. The dispersion relation for electrostatic mode can be obtained from equation (4'):

$$K_{xx}=0. \tag{9}$$

From (3) and (9), we have

$$1+2\pi\sum_j\left(\frac{\Omega_j\pi_j}{k\omega}\right)^2\sum_{\ell=-\infty}^{\infty}\ell^2\int_0^\infty J_\ell^2\left(\frac{kv_\perp}{\Omega_j}\right)dv_\perp\int_{-\infty}^\infty\frac{\omega}{\omega-\ell\Omega_j}\frac{\partial f_{oj}}{\partial v_\perp}dv_\parallel=0. \tag{9'}$$

In this electrostatic mode the electric wave field is perpendicular to the magnetic field $\mathbf{B}_0$ and is parallel to the propagation vector $\mathbf{k}$.

III. DISPERSION RELATIONS FOR A BI-MAXWELLIAN MAGNETOPLASMA

Let the zero-order velocity distribution function of each "j" charge species in the collisionless uniform magnetoplasma be of the form

$$f_{oj}(v_\perp, v_\parallel) = \pi^{-3/2}v_{j\perp}^{-2}v_{j\parallel}^{-1}e^{-(v_\perp^2/v_{j\perp}^2+v_\parallel^2/v_{j\parallel}^2)}, \tag{10}$$

where v_j refers to thermal speed of the "j" charge species. Expression (10) is a solution of the zero-order Vlasov equation@). Experimental measurements from space probe+ have indicated that the distribution functions of solar wind plasma are of the form (10). Cefine the thermal anisotropy of the "j" charge species by

(5) R. W. Landau and S. Cuperman, J. Plasma Phys. 4, 13, (1970).

(6) N.G. Van Kampen and B.U. Felderhof, ***Theoretical Methods in Plasma Physics,*** pp. 178-187, (North-Holland, Amsterdam, 1967).

(7) A. J. Hundhausen, J. R. Asbridge, S. J. Bame, H. E. Gilbert, and I. B. Strong, J. Geophys. Res. 72, 87, (1967).

$$\delta_j \equiv 1 - \frac{v_{j\parallel}^2}{v_{j\perp}^2} = 1 - \frac{T_{j\parallel}}{T_{j\perp}}, \tag{11}$$

where T_j is the temperature of the "j" charge species, $\parallel$ and $\perp$ refer to parallel and perpendicular with respect to B_0, and

$$v_{j\parallel} \equiv \left(\frac{2\kappa T_{j\parallel}}{m_j}\right)^{1/2}, \qquad v_{j\perp} \equiv \left(\frac{2\kappa T_{j\perp}}{m_j}\right)^{1/2}.$$

κ is the Boltzmann's constant. Also put

$$\alpha_{j\ell} \equiv \frac{\omega + \ell\Omega_j}{k_\parallel v_{j\parallel}} \qquad b_j \equiv \frac{1}{2}\left(\frac{k_\perp v_{j\perp}}{\Omega_j}\right)^2. \tag{12}$$

Using (10), (11), and (12) in (3), we are able to obtain the dielectric tensor for the Bi-Maxwellian magnetoplasma in terms of the plasma dispersion function $Z(\alpha_{j\ell})$ defined by Fried and Conte[4], the modified Bessel function $I_\ell(b_j)$ and its derivative $I'_\ell(b_j)$:

$$K_{xx} = 1 + \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{e^{-b_j}}{(1-\delta_j)b_j} \sum_{\ell=-\infty}^{\infty} \ell^2 \left[\delta_j + \frac{\omega + \ell\delta_j\Omega_j}{\omega + \ell\Omega_j}\alpha_{j\ell}Z(\alpha_{j\ell})\right] I_\ell(b_j),$$

$$K_{yy} = 1 + \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{e^{-b_j}}{1-\delta_j} \sum_{\ell=-\infty}^{\infty} \left[\delta_j + \frac{\omega + \ell\delta_j\Omega_j}{\omega + \ell\Omega_j}\alpha_{j\ell}Z(\alpha_{j\ell})\right]\left[\frac{\ell^2}{b_j} I_\ell(b_j) + 2b_j I_\ell(b_j) - 2b_j I'_\ell(b_j)\right],$$

$$K_{zz} = 1 + 2\sum_j \left(\frac{\pi_j}{\omega}\right)^2 e^{-b_j} \sum_{\ell=-\infty}^{\infty} \alpha_{j\ell}^2 [1 + \alpha_{j\ell}Z(\alpha_{j\ell})] I_\ell(b_j) \frac{\omega + \ell\delta_j\Omega_j}{\omega + \ell\Omega_j},$$

$$K_{xy} = i\sum_j \epsilon_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{e^{-b_j}}{1-\delta_j} \sum_{\ell=-\infty}^{\infty} \ell \left[\delta_j + \frac{\omega + \ell\delta_j\Omega_j}{\omega + \ell\Omega_j}\alpha_{j\ell}Z(\alpha_{j\ell})\right][I_\ell(b_j) - I'_\ell(b_j)] = -K_{yx},$$

$$K_{xz} = -\sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{k_\perp e^{-b_j}}{k_\parallel \Omega_j (1-\delta_j) b_j} \sum_{\ell=-\infty}^{\infty} \ell(\omega + \ell\delta_j\Omega_j)[1 + \alpha_{j\ell}Z(\alpha_{j\ell})] I_\ell(b_j) = K_{zx},$$

$$K_{yz} = i\sum_j \epsilon_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{k_\perp e^{-b_j}}{k_\parallel \Omega_j (1-\delta_j)} \sum_{\ell=-\infty}^{\infty} (\omega + \ell\delta_j\Omega_j)[1 + \alpha_{j\ell}Z(\alpha_{j\ell})][I_\ell(b_j) - I'_\ell(b_j)] = -K_{zy}. \tag{13}$$

The details of derivation are given in Appendix A. The summation in j is carried over all charge species. By putting (13) into (4′) and using (11), (12) also the relation $\sum_{\ell=-\infty}^{\infty} \ell I_\ell(b_j) = 0$, the dispersion relation for electrostatic mode becomes

$$1 + 2\sum_j \left(\frac{\pi_j}{k v_{j\parallel}}\right)^2 \left[1 + e^{-b_j} \sum_{\ell=-\infty}^{\infty} \frac{\omega + \ell\delta_j\Omega_j}{\omega + \ell\Omega_j}\alpha_{j\ell}Z(\alpha_{j\ell}) I_\ell(b_j)\right] = 0. \tag{14}$$

For parallel and perpendicular propagation with respect to the magnetic field, the dispersion relations can be further simplified.

1. **Parallel Propagation**

For propagation along the magnetic field $\mathbf{B}_0$, we have

$$k = k_\parallel, \; k_\perp = 0, \; b_j = 0, \; I_\ell(b_j) = 0 \quad \text{for} \quad \ell \neq 0, \; I_0(0) = 1, \; \alpha_{j0} = \frac{\omega}{k v_{j\parallel}}. \tag{15}$$

The dispersion relation for electrostatic mode in (14) becomes

$$1 + 2\sum_j \left(\frac{\pi_j}{k v_{j\parallel}}\right)^2 [1 + \alpha_{jo} Z(\alpha_{jo})] = 0. \tag{16}$$

Putting (13) into (6) and also using (15), the dispersion relation for cyclotron waves becomes

$$\left(\frac{kc}{\omega}\right)^2 = 1 + \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{1}{1-\delta_j}\left[\delta_j + \frac{\omega \pm \epsilon_j\Omega_j\delta_j}{\omega \pm \epsilon_j\Omega_j}\alpha_j Z(\alpha_j)\right]. \left\}\begin{matrix}\text{RH}\\ \text{LH.}\end{matrix}\right. \tag{17}$$

where $\alpha_j = \frac{\omega \pm \epsilon_j \Omega_j}{k v_{j\parallel}}$, upper sign is for RH cyclotron waves and lower sign is for LH cyclotron waves.

2. Perpendicular Propagation

For propagation across the magnetic field B,, we have

$$k = k_\perp, \qquad k_\parallel = 0, \qquad \alpha_{j\ell} Z(\alpha_{j\ell}) = -1. \tag{18}$$

The dispersion relation for electrostatic mode in (14) becomes

$$1 - 2\sum_j \pi_j^2 \frac{e^{-b_j}}{b_j} \sum_{\ell=1}^{\infty} \frac{\ell^2 I_\ell(b_j)}{\omega^2 - \ell^2 \Omega_j^2} = 0. \tag{19}$$

Putting (13) into (7) and also using (18), the dispersion relation for transverse mode (ordinary mode) becomes

$$\left(\frac{kc}{\omega}\right)^2 = 1 - \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \left[1 + 2e^{-b_j} \sum_{\ell=1}^{\infty} \frac{(1-\delta_j)\ell^2 \Omega_j^2}{\omega^2 - \ell^2 \Omega_j^2} I_\ell(b_j)\right]. \tag{20}$$

From (13), (18) and (8), the dispersion relation for hybrid mode (extrabrdinary mode) becomes

$$\left(\frac{kc}{\omega}\right)^2 = K_{yy} + \frac{K_{xy}^2}{K_{xx}}, \tag{21}$$

where

$$K_{xx} = 1 - 2\sum_j \pi_j^2 \frac{e^{-b_j}}{b_j} \sum_{\ell=1}^{\infty} \frac{\ell^2 I_\ell(b_j)}{\omega^2 - \ell^2 \Omega_j^2},$$

$$K_{yy} = 1 - \sum_j \frac{\pi_j^2}{\omega} e^{-b_j} \sum_{\ell=-\infty}^{\infty} \left[\frac{\ell^2}{b_j} I_\ell(b_j) + 2b_j I_\ell(b_j) - 2b_j I_\ell'(b_j)\right] / (\omega + \ell \Omega_j).$$

$$K_{xy} = -i \sum_j \epsilon_j \frac{\pi_j^2}{\omega} e^{-b_j} \sum_{\ell=-\infty}^{\infty} \ell [I_\ell(b_j) - I_\ell'(b_j)] / (\omega + \ell \Omega_j).$$

APPENDIX A

To obtain the expressions of the dielectric tensor in (13), we use the Weber's second exponentiul integral[8,9]:

$$\int_0^\infty x e^{-\rho^2 x^2} J_\ell(\beta x) J_\ell(\gamma x) dx = \frac{1}{2\rho^2} e^{-(\beta^2+\gamma^2)/4\rho^2} I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right). \tag{A1}$$

When $\beta = \gamma$, (Al) becomes

$$\int_0^\infty x e^{-\rho^2 x^2} J_\ell^2(\beta x) dx = \frac{1}{2\rho^2} e^{-\beta^2/2\rho^2} I_\ell\left(\frac{\beta^2}{2\rho^2}\right). \tag{A2}$$

Putting $x = v_\perp, \beta = k_\perp/\Omega_j, \rho = v_{j\perp}^{-1}, b_j = \beta^2/2\rho^2$, (A2) becomes

$$\int_0^\infty v_\perp e^{-v_\perp^2/v_{j\perp}^2} J_\ell^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right) dv_\perp = \frac{1}{2} v_{j\perp}^2 e^{-b_j} I_\ell(b_j). \tag{A3}$$

Differentiating (10) with respect to $v_\perp$ and $v_\parallel$, we have

(8) G.N. Watson, ***A Treatise on the Theory of Bessel Functions***, 2nd ed., p. 395, (Cambridge University Press, Cambridge, 1958).

(9) I. S. Gradshteyn and I. M. Ryzhik, ***Table of Integrals,*** *Series and Products,* p. 718, (Academic Press, New York, 1965).

$$\frac{\partial f_{oj}}{\partial v_\perp} = -2\pi^{-3/2} v_{j\perp}^{-4} v_{j\parallel}^{-1} v_\perp e^{-(v_\perp^2/v_{j\perp}^2 + v_\parallel^2/v_{j\parallel}^2)}, \tag{A4}$$

and

$$\frac{\partial f_{oj}}{\partial v_\parallel} = -2\pi^{-3/2} v_{j\parallel}^{-3} v_{j\perp}^{-2} v_\parallel e^{-(v_\perp^2/v_{j\perp}^2 + v_\parallel^2/v_{j\parallel}^2)}. \tag{A5}$$

The first expression in (3) is

$$K_{xx} = 1 + 2\pi \sum_j \left(\frac{\Omega_j \pi_j}{k_\perp \omega}\right)^2 \sum_{\ell=-\infty}^{\infty} \ell^2 \int_0^\infty J_\ell^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right) dv_\perp \int_{-\infty}^{\infty} \frac{(\omega - k_\parallel v_\parallel)\dfrac{\partial f_{oj}}{\partial v_\perp} + k_\parallel v_\perp \dfrac{\partial f_{oj}}{\partial v_\parallel}}{\omega - k_\parallel v_\parallel - \ell\Omega_j} dv_\parallel \tag{A6}$$

The integration over $v_\perp$ can be first carried out immediately by putting (A4) and (A5) into (A6) and using (A3). After integration over $v_\perp$, (A6) becomes

$$K_{xx} = 1 - 2\pi^{-1/2} \sum_j \left(\frac{\Omega_j \pi_j}{k_\perp \omega}\right)^2 v_{j\perp}^2 e^{-b_j} \sum_{\ell=-\infty}^{\infty} \ell^2 I_\ell(b_j)$$
$$\int_{-\infty}^{\infty} \left[\frac{v_{j\perp}^{-4} v_{j\parallel}^{-1}(\omega - k_\parallel v_\parallel) + v_{j\perp}^{-2} v_{j\parallel}^{-3} k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel - \ell\Omega_j}\right] e^{-v_\parallel^2/v_{j\parallel}^2} dv_\parallel. \tag{A7}$$

Rearrange the square bracket in the integrand of (A7) as follows:

$$\frac{v_{j\perp}^{-4} v_{j\parallel}^{-1}(\omega - k_\parallel v_\parallel) + v_{j\perp}^{-2} v_{j\parallel}^{-3} k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel - \ell\Omega_j}$$
$$= \frac{v_{j\perp}^{-4} v_{j\parallel}^{-1}[(\omega - k_\parallel v_\parallel - \ell\Omega_j) + \ell\Omega_j] + v_{j\perp}^{-2} v_{j\parallel}^{-3}[(k_\parallel v_\parallel + \ell\Omega_j - \omega) - \ell\Omega_j + \omega]}{\omega - k_\parallel v_\parallel - \ell\Omega_j}$$
$$= v_{j\perp}^{-4} v_{j\parallel}^{-1} - v_{j\perp}^{-2} v_{j\parallel}^{-3} + \frac{\ell\Omega_j(v_{j\perp}^{-4} v_{j\parallel}^{-1} - v_{j\perp}^{-2} v_{j\parallel}^{-3}) + v_{j\perp}^{-2} v_{j\parallel}^{-3}\omega}{\omega - k_\parallel v_\parallel - \ell\Omega_j}$$
$$= -v_{j\perp}^{-2} v_{j\parallel}^{-3}\left[(1 - v_{j\parallel}^2/v_{j\perp}^2) + \frac{\ell\Omega_j(1 - v_{j\parallel}^2/v_{j\perp}^2) - \omega}{-k_\parallel\left(v_\parallel - \dfrac{\omega - \ell\Omega_j}{k_\parallel}\right)}\right]$$
$$= -v_{j\perp}^{-2} v_{j\parallel}^{-3}\left[\delta_j + \frac{\omega - \ell\Omega_j\delta_j}{k_\parallel\left(v_\parallel - \dfrac{\omega - \ell\Omega_j}{k_\parallel}\right)}\right], \tag{A8}$$

where (11) has been used. To carry out the integration over $v_\parallel$ in (A7), we use the plasma dispersion function Z defined by Fried and Conte[4]:

$$Z(\alpha) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-y^2} dy}{y - \alpha} \tag{A9}$$

for Im $\alpha > 0$ and its analaytic continuation for Im $\alpha \leq 0$. From (11) and (12), we have

$$\frac{2\Omega_j^2}{k_\perp^2} = \frac{v_{j\parallel}^2}{b_j(1 - \delta_j)}. \tag{A10}$$

Putting (A8) and (A10) into (A7), we have

$$K_{xx} = 1 + \pi^{-1/2} \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{e^{-b_j}}{b_j(1-\delta_j)} \sum_{\ell=-\infty}^{\infty} \ell^2 I_\ell(b_j) \int_{-\infty}^{\infty} \left[\frac{\delta_j}{v_{j\parallel}} + \frac{\omega - \ell\delta_j\Omega_j}{k_\parallel v_{j\parallel}\left(v_\parallel - \dfrac{\omega - \ell\Omega_j}{k_\parallel}\right)}\right] e^{-v_\parallel^2/v_{j\parallel}^2} dv_\parallel$$
$$= 1 + \sum_j \left(\frac{\pi_j}{\omega}\right)^2 \frac{e^{-b_j}}{(1-\delta_j)b_j} \sum_{\ell=\infty}^{-\infty} \ell^2 I_{-\ell}(b_j)\left[\delta_j + \pi^{-1/2}\frac{\omega + \ell\delta_j\Omega_j}{k_\parallel v_{j\parallel}} \int_{-\infty}^{\infty} \frac{e^{-v_\parallel^2/v_{j\parallel}^2}}{v_\parallel - \dfrac{\omega + \ell\Omega_j}{k_\parallel}} dv_\parallel\right]$$

$$=1+\sum_j\left(\frac{\pi_j}{\omega}\right)^2\frac{e^{-b_j}}{(1-\delta_j)b_j}\sum_{\ell=-\infty}^{\infty}\ell^2I_\ell(b_j)\left[\delta_j+\pi^{-1/2}\frac{\omega+\ell\delta_j\Omega_j}{k_\parallel v_{j\parallel}}\int_{-\infty}^{\infty}\frac{e^{-v_\parallel^2/v^2_{\parallel}}}{\frac{v_\parallel}{v_{j\parallel}}-\frac{\omega+\ell\Omega_j}{k_\parallel v_{j\parallel}}}d\left(\frac{v_\parallel}{v_{j\parallel}}\right)\right]$$

$$=1+\sum_j\left(\frac{\pi_j}{\omega}\right)^2\frac{e^{-bj}}{(1-\delta_j)b_j}\sum_{\ell=-\infty}^{\infty}\ell^2I_\ell(b_j)\left[\delta_j+\frac{\omega+\ell\delta_j\Omega_j}{k_\parallel v_{j\parallel}}Z\left(\frac{\omega+\ell\Omega_j}{k_\parallel v_{j\parallel}}\right)\right]$$

$$=1+\sum_j\left(\frac{\pi_j}{\omega}\right)^2\frac{e^{-bj}}{(1-\delta_j)b_j}\sum_{\ell=-\infty}^{\infty}\ell^2I_\ell(b_j)\left[\delta_j+\frac{\omega+\ell\delta_j\Omega_j}{\omega+\ell\Omega_j}\alpha_{j\ell}Z(\alpha_{j\ell})\right], \tag{A11}$$

where $I_\ell(b_j)=I_{-\ell}(b_j)$, also (A9) and (12) have been used. The expression (A11) is the first expression in (13). To obtain the second expression in (13), we need also the differentiation of (A1) with respect to β and γ.

$$\because\quad \frac{dJ_\ell(\beta x)}{d\beta}=\frac{dJ_\ell(\beta x)}{d(\beta x)}\frac{d(\beta x)}{d\beta}=xJ_\ell'(\beta x),$$

also

$$\frac{dJ_\ell(\gamma x)}{d\gamma}=xJ_\ell'(\gamma x),$$

$$\frac{dI_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)}{d\beta}=\frac{\gamma}{2\rho^2}I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right),$$

$$\frac{dI_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)}{d\gamma}=\frac{\beta}{2\rho^2}I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right).$$

Differentiating (A1) with respect to β, we have

$$\int_0^\infty x^2e^{-\rho^2x^2}J_\ell'(\beta x)J_\ell(\gamma x)dx=\frac{e^{-(\beta^2+\gamma^2)/4\rho^2}}{4\rho^4}\left[\gamma I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right)-\beta I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)\right]. \tag{A12}$$

Differentiating (A12) with respect to γ, we have

$$\int_0^\infty x^3e^{-\rho^2x^2}J_\ell'(\beta x)J_\ell'(\gamma x)dx$$
$$=\frac{e^{-(\beta^2+\gamma^2)/4\rho^2}}{8\rho^6}\left[\beta\gamma I_\ell''\left(\frac{\beta\gamma}{2\rho^2}\right)+(2\rho^2-\gamma^2-\beta^2)I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right)+\beta\gamma I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)\right]. \tag{A13}$$

Using recurrence relations for the modified Bessel functions[10], we have

$$I_\ell''(y)=\left(y+\frac{\ell^2}{y}\right)I_\ell(y)-I_\ell'(y),$$

or

$$\beta\gamma I_\ell''\left(\frac{\beta\gamma}{2\rho^2}\right)=\left(\beta\gamma+\frac{4\rho^4\ell^2}{\beta\gamma}\right)I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)-2\rho^2I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right). \tag{A14}$$

Putting (A14) into (A13), we have

$$\int_0^\infty x^3e^{-\rho^2x^2}J_\ell'(\beta x)J_\ell'(\gamma x)dx$$
$$=\frac{e^{-(\beta^2+\gamma^2)/4\rho^2}}{8\rho^6}\left[2\beta\gamma I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)+\frac{4\rho^4\ell^2}{\beta\gamma}I_\ell\left(\frac{\beta\gamma}{2\rho^2}\right)-(\beta^2+\gamma^2)I_\ell'\left(\frac{\beta\gamma}{2\rho^2}\right)\right]. \tag{A15}$$

(10) M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, p. 139, (McGraw-Hill, New York, 1968).

Putting $\beta=\gamma$ in (A12) and (A15), we have

$$\int_0^\infty x^2 e^{-\rho^2 x^2} J_\ell'(\beta x) J_\ell(\beta x)\,dx = \frac{\beta e^{-\beta^2/2\rho^2}}{4\rho^4}\left[I_\ell'\left(\frac{\beta^2}{2\rho^2}\right) - I_\ell\left(\frac{\beta^2}{2\rho^2}\right)\right], \tag{A16}$$

$$\int_0^\infty x^3 e^{-\rho^2 x^2} J_\ell'^2(\beta x)\,dx = \frac{e^{-\beta^2/2\rho^2}}{8\rho^6}\left[2\beta^2 I_\ell\left(\frac{\beta^2}{2\rho^2}\right) + \frac{4\rho^4 \ell^2}{\beta^2} I_\ell\left(\frac{\beta^2}{2\rho^2}\right) - 2\beta^2 I_\ell'\left(\frac{\beta^2}{2\rho^2}\right)\right]. \tag{A17}$$

Again putting $x=v_\perp, b_j=\dfrac{\beta^2}{2\rho^2}, \beta=k_\perp/\Omega_j, \rho=v_{j\perp}^{-1}$ into (A16) and (A17), we have

$$\int_0^\infty v_\perp^2 e^{-v_\perp^2/v_{j\perp}^2} J_\ell'\left(\frac{k_\perp v_\perp}{\Omega_j}\right) J_\ell\left(\frac{k_\perp v_\perp}{\Omega_j}\right) dv_\perp = \frac{k_\perp v_{j\perp}^4 e^{-b_j}}{4\Omega_j}[I_\ell'(b_j) - I_\ell(b_j)], \tag{A18}$$

$$\int_0^\infty v_\perp^3 e^{-v_\perp^2/v_{j\perp}^2} J_\ell'^2\left(\frac{k_\perp v_\perp}{\Omega_j}\right) dv_\perp = \frac{v_{j\perp}^4 e^{-b_j}}{4}\left[2b_j I_\ell(b_j) + \frac{\ell^2}{b_j} I_\ell(b_j) - 2b_j I_\ell'(b_j)\right]. \tag{A19}$$

Put (A4) and (A5) into the second expression in (3) and integrate first over $v_\perp$ by using (A19) and then integrate over $v_\parallel$ by using (A9), we obtain the second expression in (13) similar to the method given to derive (All). The other expresions in (13) can be derived similarly by using (A3) or (A18) for integration over $v_\perp$, and (A9) for integration over $v_\parallel$.