

The Bounds for Truncation Error in Sampling Expansions

JYI CHERNG HORNG*

(Received 7 March 1975; revised manuscript received 25 December 1975)

The bounds for truncation error in sampling expansions with sampled zero-th, first and second order derivatives are studied. The result shows that the upper bounds for truncation error ($\epsilon_N(z)$, $\epsilon'_N(z)$, $\epsilon''_N(z)$) decrease respectively as sampling points (N) increase if β and k are constants. In addition to the result mentioned above the upper bound for truncation error with sampled zero-th and first order derivatives is about

$$\{\sin(\beta z) | (1 + 2(E'_k/E_k)) / (\sqrt{3}(N\pi - |\beta z|))\}$$

times as large as the upper bound for the truncation error with sampled zero-th order derivatives.

I. INTRODUCTION

The bounds for truncation error of a sampling expansion were treated by many authors; Takizawa,^(1,2) Helms and Thomas,⁽³⁾ Papoulis,^(4,5) Brown,⁽⁶⁾ and Jagerman.⁽⁷⁾ A bound for truncation error of Shannon's sampling expansion with sampled zero-th and first order derivatives was investigated by Helms and Thomas.⁽³⁾ Recently, a bound for truncation error of the generalized sampling expansion with sampled zero-th, first, and second order derivatives was mentioned by Takizawa.^(1,2)

In this paper, we shall discuss more precisely the truncation error of sampling expansions (with sampled higher order derivatives), in which the successive sampling points are equally spaced.

II. TRUNCATION ERROR IN THE SAMPLING EXPANSION WITH SAMPLED ZERO-TH ORDER DERIVATIVES

The sampling expansion for an entire function $f(z)$ reads as follows⁽¹⁾:

$$f(z) = \sum_{n=-\infty}^{+\infty} f_n \cdot \frac{g(z)}{(z - \lambda_n) \cdot g'_n} + K \cdot g(z), \quad (1)$$

with $f_n = f(\lambda_n)$, $g'_n = g'(\lambda_n)$, $K = \text{const}$. Sampling points λ_n ($n = \text{integers}$) are simple zeros of an entire function $g(z)$. z and λ_n are taken to be real. The summation over n means to cover all the sampling points.

We shall consider the truncation error $\epsilon_n(z)$ of sampling expansion (1) defined by:

* 洪吉成, Visiting Research Associate of Hokkaidô University, now on leave from the Physics Section, Tatung Institute of Technology (大同工學院), Taipei, Taiwan, China.

(1) È. I. Takizawa, S.M. Hsieh and J.C. Horng: Chinese J. Phys. **13**, **87**. (1975),

(2) È. I. Takizawa: Chinese J. Phys. **13**, 226 (1975).

(3) H. D. Helms and J.B. Thomas: Proc. IRE **50**, 179 (1962).

(4) A. Papoulis: Proc. IRE **54**, 947 (1966).

(5) A. Papoulis: IEEE Trans. on Automatic Control AC-12, 604 (1967).

(6) J. L. Brown: IEEE Trans. on Information Theory IT-15, 440 (1969).

(7) D. Jagerman: J. SIAM Appl. Math. **14**, 714 (1966).

$$\varepsilon_N(z) \equiv f(z) - \sum_{n=-N}^N f_n \cdot \frac{g(z)}{(z-\lambda_n) \cdot g'_n} - K \cdot g(z). \quad (2)$$

Here N is a positive integer. From (1) and (2), we have:

$$\varepsilon_N(z) = \sum_{n>N} f_n \cdot \frac{g(z)}{(z-\lambda_n) \cdot g'_n} + \sum_{n<-N} f_n \cdot \frac{g(z)}{(z-\lambda_n) \cdot g'_n} \quad (\lambda_{-N} < z < \lambda_N). \quad (3)$$

Now we shall put $g(z) = \sin(\beta z)$ ($\beta > 0$) in (1)~(3), and we obtain sampling points $\lambda_n = n\pi/\beta$ (n -integers).

Applying Cauchy's inequality to (3), we have:

$$|\varepsilon_N(z)| \leq |\sin(\beta z)| \cdot \left[\left\{ \sum_{n>N} |f_n|^2 \cdot \sum_{n>N} \left| \frac{1}{\beta z - n\pi} \right|^2 \right\}^{1/2} + \left\{ \sum_{n<-N} |f_n|^2 \cdot \sum_{n<-N} \left| \frac{1}{\beta z - n\pi} \right|^2 \right\}^{1/2} \right], \quad (4)$$

with $|g'_n| = \beta$.

We shall mention that $1/(\beta z - u)^2$ ($\beta > 0$) is a monotonic decreasing function of u for $u > \beta z$. Hence we obtain:

$$\sum_{n>N} \frac{1}{(\beta z - n\pi)^2} < \frac{1}{\pi} \int_{N\pi}^{\infty} \frac{du}{(\beta z - u)^2} = \frac{1}{\pi(N\pi - \beta z)}. \quad (\beta z < N\pi) \quad (5)$$

From (5) we have:

$$\left\{ \sum_{n>N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} < \frac{1}{\sqrt{\pi(N\pi - \beta z)}}; \quad (\beta z < N\pi) \quad (6)$$

Similarly, we obtain:

$$\left\{ \sum_{n<-N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} < \frac{1}{\sqrt{\pi(\beta z + N\pi)}}. \quad (-N\pi < \beta z) \quad (7)$$

Putting (6) and (7) into (4), we have:

$$|\varepsilon_N(z)| \leq \frac{|\sin(\beta z)|}{\sqrt{\pi}} \cdot \left[\frac{K_N}{\sqrt{N\pi - \beta z}} + \frac{L_N}{\sqrt{\beta z + N\pi}} \right], \quad (-N\pi/\beta < z < N\pi/\beta) \quad (8)$$

with

$$K_N = \left\{ \sum_{n>N} |f_n|^2 \right\}^{1/2}, \text{ and } L_N = \left\{ \sum_{n<-N} |f_n|^2 \right\}^{1/2}. \quad (9)$$

In order to estimate $|\varepsilon_N(z)|$, we shall assume that

$$E_k \equiv \left\{ \int_{-\infty}^{+\infty} z^{2k} \cdot |f(z)|^2 dz \right\}^{1/2} < +\infty, \quad (10)$$

Here k is a positive integer. Function $f(z)$ is entire, thus $z^k \cdot f(z)$ is also entire. Hence we can write^(8,9),

$$z^k \cdot f(z) = a^k \cdot \sum_{n=-\infty}^{+\infty} n^k \cdot f_n \cdot \frac{\sin(\beta z + \gamma - n\pi)}{\beta z + \gamma - n\pi}, \quad (11)$$

with $a = \pi/\beta$.

Functions $\{\sin(\beta z + \gamma - n\pi)/(\beta z + \gamma - n\pi): n = \text{integers}\}$ are an orthogonal system in the interval $-\infty < z < +\infty$, i.e.

$$\int_{-\infty}^{\infty} \frac{\sin(\beta z + \gamma - m\pi)}{\beta z + \gamma - m\pi} \cdot \frac{\sin(\beta z + \gamma - n\pi)}{\beta z + \gamma - n\pi} dz = \frac{\pi}{\beta} \cdot \delta_{m,n}. \quad (12)$$

Hence we obtain the following expression from (10) and (11):

(8) C. E. Shannon: Bell System Tech. J. 27, 379 and 623 (1948).

(9) C. E. Shannon and W. Weaver: *Mathematical Theory of Communication* (Univ. of Illinois Press, 1949).

$$E_k^2 = \int_{-\infty}^{+\infty} z^{2k} \cdot |f(z)|^2 dz = a^{2k+1} \cdot \sum_{n=-\infty}^{+\infty} (n)^{2k} \cdot |f_n|^2. \quad (13)$$

Accordingly we have:

$$a^{2k+1} \cdot \sum_{n=m}^{2m-1} (n)^{2k} \cdot |f_n|^2 \leq E_k^2, \quad (m = \text{positive integer}) \quad (14)$$

i. e.

$$\sum_{n=m}^{2m-1} |f_n|^2 \leq \frac{E_k^2}{a^{2k+1} \cdot m^{2k}}. \quad (15)$$

We calculate from (9)

$$K_N^2 = \sum_{n>N} |f_n|^2 = \sum_{n=N+1}^{\infty} |f_n|^2 = \sum_{\alpha=0}^{\infty} \sum_{n=m}^{2m-1} |f_n|^2, \quad (16)$$

with $m = 2^\alpha \cdot (N + 1)$. ($\alpha =$ non-negative integers, $N =$ a positive integer) Putting (15) into (16), we obtain :

$$K_N^2 \leq \sum_{\alpha=0}^{\infty} \frac{E_k^2}{a^{2k+1} \cdot \{2^\alpha \cdot (N+1)\}^{2k}} = \frac{E_k^2}{a^{2k+1} \cdot (N+1)^{2k}} \cdot \sum_{\alpha=0}^{\infty} 4^{-\alpha k}, \quad (17)$$

hence, we have:

$$K_N \leq \frac{E_k}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}. \quad (18)$$

Similarly, we obtain:

$$L_N \leq \frac{E_k}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}. \quad (19)$$

Putting (18) and (19) into (8), we have:

$$|\varepsilon_N(z)| \leq \frac{|\sin(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{E_k}{a^{k+(1/2)} \cdot (N+1)^k \cdot \sqrt{1-4^{-k}}} \right\} \cdot \left\{ \frac{1}{\sqrt{N\pi - \beta z}} + \frac{1}{\sqrt{\beta z + N\pi}} \right\}. \quad (-N\pi/\beta < z < N\pi/\beta) \quad (20)$$

From (20) we obtain:

$$|\varepsilon_N(z)| \leq \frac{|\sin(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{E_k}{a^{k+(1/2)} \cdot (N+1)^k \cdot \sqrt{1-4^{-k}}} \right\} \cdot \frac{2}{\sqrt{N\pi - |\beta z|}}. \quad (21)$$

As a special case, we put $z = \pi/(2\beta)$ into (21), and we have:

$$|\varepsilon_N(\pi/2\beta)| \leq \frac{1}{\pi} \left(\frac{\beta}{\pi} \right)^{k+(1/2)} \cdot \frac{E_k}{(N+1)^k \cdot \sqrt{1-4^{-k}}} \cdot \frac{2}{\sqrt{N-(1/2)}}. \quad (22)$$

As for the numerical examples, we shall take constants β , N , and k , to be such values as listed in **Table I**. The values of the right-hand side of (22) are listed in the last column. These values decrease when N increases.

Table Z

β	N	k	The values of the right-hand side of (22)
6	4	1	0.207 E_1
6	6	1	0.118 E_1
6	12	1	0.044 E_1
6	24	1	0.016 E_1

III. TRUNCATION ERROR IN THE SAMPLING EXPANSION WITH SAMPLED ZERO-TH AND FIRST ORDER DERIVATIVES

A sampling formula which includes sampled zero-th and first order derivatives of an entire function $f(z)$ is expressed by⁽¹⁾:

$$f(z) = \sum_{n=-\infty}^{+\infty} \left[f_n + (z - \lambda_n) \left\{ f'_n - \frac{1}{3} f_n \frac{g_n^{(3)}}{g_n^{(2)}} \right\} \right] \cdot \frac{2! g(z)}{(z - \lambda_n)^2 \cdot g_n^{(2)}} + K \cdot g(z), \quad (23)$$

with $f_n^{(p)} = f^{(p)}(\lambda_n)$, $g_n^{(p)} = g^{(p)}(\lambda_n)$, $K = \text{const}$. Sampling points λ_n ($n = \text{integers}$) are second order zeros of an entire function $g(z)$. The summation over n means to cover all the sampling points.

We shall consider the truncation error in the sampling expansion (23) for real z and λ_n ($n = \text{integers}$). Let $\epsilon'_N(z)$ be a truncation error defined by:

$$\begin{aligned} \epsilon'_N(z) \equiv & \sum_{n > N} \left[f_n + (z - \lambda_n) \cdot \left\{ f'_n - \frac{1}{3} f_n \cdot \frac{g_n^{(3)}}{g_n^{(2)}} \right\} \right] \cdot \frac{2! g(z)}{(z - \lambda_n)^2 \cdot g_n^{(2)}} + \\ & \sum_{n < -N} \left[f_n + (z - \lambda_n) \cdot \left\{ f'_n - \frac{1}{3} f_n \cdot \frac{g_n^{(3)}}{g_n^{(2)}} \right\} \right] \cdot \frac{2! g(z)}{(z - \lambda_n)^2 \cdot g_n^{(2)}}. \quad (\lambda_{-N} < z < \lambda_N). \quad (24) \end{aligned}$$

Here N is a positive integer.

We shall put $g(z) = \sin^2(\beta z)$ in (23) and (24). Sampling points $\lambda_n = n\pi/\beta$ ($n = \text{integers}$) are zeros of second order of $g(z)$. Applying the Cauchy inequality to (24), we have:

$$\begin{aligned} |\epsilon'_N(z)| \leq & |\sin^2(\beta z)| \cdot \left[\left\{ \sum_{n > N} |f_n|^2 \cdot \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^2} \right|^2 \right\}^{1/2} \right. \\ & + \left\{ \sum_{n > N} |(z - n\pi/\beta) \cdot f'_n|^2 \cdot \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^2} \right|^2 \right\}^{1/2} + \left\{ \sum_{n < -N} |f_n|^2 \cdot \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^2} \right|^2 \right\}^{1/2} \\ & \left. + \left\{ \sum_{n < -N} |(z - n\pi/\beta) \cdot f'_n|^2 \cdot \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^2} \right|^2 \right\}^{1/2} \right], \quad (-N\pi/\beta < z < N\pi/\beta) \quad (25) \end{aligned}$$

with

$$|g_n^{(2)}| = 2\beta^2, \text{ and } |g_n^{(3)}| = 0.$$

$1/(\beta z - u)^4$ and $1/(\beta z - u)^2$ are monotonic decreasing functions of u for $u > \beta z$. Hence we obtain:

$$\left\{ \sum_{n > N} \frac{1}{(\beta z - n\pi)^4} \right\}^{1/2} < \frac{1}{\sqrt{3\pi(N\pi - \beta z)^3}}, \quad (z < N\pi/\beta) \quad (26)$$

and

$$\left\{ \sum_{n > N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} < \frac{1}{\sqrt{\pi(N\pi - \beta z)}}. \quad (27)$$

Similarly, we have:

$$\left\{ \sum_{n < -N} \frac{1}{(\beta z - n\pi)^4} \right\}^{1/2} < \frac{1}{\sqrt{3\pi(\beta z + N\pi)^3}}, \quad (-N\pi/\beta < z) \quad (28)$$

and

$$\left\{ \sum_{n < -N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} < \frac{1}{\sqrt{\pi(\beta z + N\pi)}}. \quad (29)$$

We shall take K_N , K'_N , L_N , and L'_N as expressed in (30).

$$K_N = \sqrt{\sum_{n>N} |f_n|^2}, K'_N = \sqrt{\sum_{n>N} |naf'_n|^2}, L_N = \sqrt{\sum_{n<-N} |f_n|^2}, \text{ and } L'_N = \sqrt{\sum_{n<-N} |naf'_n|^2}, \quad (30)$$

with $a = \pi/\beta$.

Putting (26)~(30) into (25), we obtain:

$$|\varepsilon'_N(z)| \leq \frac{|\sin^2(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{K_N + 2K'_N}{\sqrt{3(N\pi - \beta z)^3}} + \frac{L_N + 2L'_N}{\sqrt{3(\beta z + N\pi)^3}} \right\}, \quad (-N\pi/\beta < z < N\pi/\beta) \quad (31)$$

with

$$\sqrt{\sum_{n>N} |\{z - na\} \cdot f'_n|^2} \leq \sqrt{4 \sum_{n>N} |naf'_n|^2} = 2K'_N, \quad (z < Na) \quad (32)$$

and

$$\sqrt{\sum_{n<-N} |\{z - na\} \cdot f'_n|^2} \leq \sqrt{4 \sum_{n<-N} |naf'_n|^2} = 2L'_N, \quad (-Na < z) \quad (33)$$

$|\varepsilon'_N(z)|$ can be obtained by estimating the quantities K'_N and L'_N . Now we shall assume that

$$E'_k \equiv \left\{ \int_{-\infty}^{+\infty} z^{2k} \cdot |z \cdot f'(z)|^2 dz \right\}^{1/2} < +\infty. \quad (34)$$

Here k is a positive integer. Function $f(z)$ is entire, hence $f'(z)$ and $z^{k+1} \cdot f'(z)$ are also entire. Thus we can write^(2, 3, 9):

$$z^k \cdot (z \cdot f'(z)) = a^k \cdot \sum_{n=-\infty}^{+\infty} (n)^k \cdot (na \cdot f'_n) \cdot \frac{\sin(\beta z + \gamma - n\pi)}{\beta z + \gamma - n\pi}. \quad (35)$$

From (12), (34), and (35) we have:

$$(E'_k)^2 = \int_{-\infty}^{+\infty} z^{2k} \cdot |z \cdot f'(z)|^2 dz = a^{2k+1} \cdot \sum_{n=-\infty}^{+\infty} (n)^{2k} \cdot |na \cdot f'_n|^2. \quad (36)$$

From (36) and (30), we make calculations similar to (14)~(17) and obtain:

$$K'_N \leq \frac{E'_k}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}; \quad (37)$$

Similarly, we have:

$$L'_N \leq \frac{E'_k}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}. \quad (38)$$

Putting (18), (19), (37), and (38) into (31), we obtain:

$$|\varepsilon'_N(z)| \leq \frac{|\sin^2(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{1}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}} \right\} \cdot \left\{ \frac{E_k + 2E'_k}{\sqrt{3(N\pi - \beta z)^3}} + \frac{E_k + 2E'_k}{\sqrt{3(\beta z + N\pi)^3}} \right\}, \quad (|z| < N\pi/\beta) \quad (39)$$

From (39) we have:

$$|\varepsilon'_N(z)| \leq \frac{|\sin^2(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{2}{a^{k+(1/2)} \cdot (N+1)^k \cdot \sqrt{1-4^{-k}}} \right\} \cdot \frac{E_k + 2E'_k}{\sqrt{3(N\pi - |\beta z|)^3}}, \quad (40)$$

As a special case, we put $z = \pi/(2\beta)$ into (40), and we obtain:

$$|\varepsilon'_N(\pi/2\beta)| \leq \frac{2}{\pi^2} \cdot \left(\frac{\beta}{\pi} \right)^{k+(1/2)} \cdot \frac{1}{(N+1)^k \cdot \sqrt{1-4^{-k}}} \cdot \frac{E_k + 2E'_k}{\sqrt{3(N - (1/2))^3}}. \quad (41)$$

When we compare the right-hand side of (40), with the same side of (21), we can see the value of the right-hand side of (40) is about $|\sin(\beta z)| \cdot (1 + 2(E_k/E_k)) / \{\sqrt{3}(N\pi - |\beta z|)\}$ times as large as the value of the right-hand side of (21).

As for the numerical examples in (41), we take constants β , N , and k , to be such values as listed in Table II. The values of the right-hand side of (41) are listed in the last column. These values decrease as N increases.

Table II

β	N	k	The values of the right-hand side of (41)
6	4	1	$1.1 \times 10^{-2} \cdot (E_1 + 2E_1')$
6	6	1	$3.9 \times 10^{-3} \cdot (E_1 + 2E_1')$
6	12	1	$7.0 \times 10^{-4} \cdot (E_1 + 2E_1')$
6	24	1	$1.2 \times 10^{-4} \cdot (E_1 + 2E_1')$

IV. TRUNCATION ERROR IN THE SAMPLING EXPANSION WITH SAMPLED ZERO-TH, FIRST, AND SECOND ORDER DERIVATIVES

The sampling expansion^(10,11) with sampled second order derivatives of an entire function $f(z)$, reads as follows:

$$f(z) = \sum_{n=-\infty}^{+\infty} \left[f_n + (z - \lambda_n) \cdot \left\{ f'_n - \frac{1}{4} f_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right\} + \frac{(z - \lambda_n)^2}{2} \cdot \left\{ f''_n - \frac{1}{2} f'_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right. \right. \\ \left. \left. + \frac{1}{2} f_n \cdot \left[\frac{1}{4} \left(\frac{g_n^{(4)}}{g_n^{(3)}} \right)^2 - \frac{1}{5} \frac{g_n^{(5)}}{g_n^{(3)}} \right] \right\} \right] \cdot \frac{3! g(z)}{(z - \lambda_n)^3 \cdot g_n^{(3)}} + K \cdot g(z), \quad (42)$$

with $f_n^{(p)} = f^{(p)}(\lambda_n)$, $g_n^{(p)} = g^{(p)}(\lambda_n)$, $K = \text{const}$. Sampling points λ_n ($n = \text{integers}$) are third order zeros of an entire function $g(z)$. The summation over n means to cover all the sampling points.

We shall consider the truncation error in the sampling expansion (42) for real z and λ_n ($n = \text{integers}$), which is defined by:

$$\varepsilon_N''(z) \equiv \sum_{n > N} \left[f_n + (z - \lambda_n) \cdot \left\{ f'_n - \frac{1}{4} f_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right\} + \frac{(z - \lambda_n)^2}{2} \cdot \left\{ f''_n - \frac{1}{2} f'_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right. \right. \\ \left. \left. + \frac{1}{2} f_n \cdot \left[\frac{1}{4} \left(\frac{g_n^{(4)}}{g_n^{(3)}} \right)^2 - \frac{1}{5} \frac{g_n^{(5)}}{g_n^{(3)}} \right] \right\} \right] \cdot \frac{3! g(z)}{(z - \lambda_n)^3 \cdot g_n^{(3)}} \\ + \sum_{n < -N} \left[f_n + (z - \lambda_n) \cdot \left\{ f'_n - \frac{1}{4} f_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right\} + \frac{(z - \lambda_n)^2}{2} \cdot \left\{ f''_n - \frac{1}{2} f'_n \cdot \frac{g_n^{(4)}}{g_n^{(3)}} \right. \right. \\ \left. \left. + \frac{1}{2} f_n \cdot \left[\frac{1}{4} \left(\frac{g_n^{(4)}}{g_n^{(3)}} \right)^2 - \frac{1}{5} \frac{g_n^{(5)}}{g_n^{(3)}} \right] \right\} \right] \cdot \frac{3! g(z)}{(z - \lambda_n)^3 \cdot g_n^{(3)}}. \quad (\lambda_{-N} < z < \lambda_N) \quad (43)$$

Here N is a positive integer.

We shall put $g(z) = \sin^3(\beta z)$ ($\beta > 0$) in (42) and (43). Sampling points $\lambda_n = n\pi/\beta$ ($n = \text{integers}$) are zeros of third order of $g(z)$.

Applying the Cauchy inequality to (43), we have:

$$|\varepsilon_N''(z)| \leq |\sin^3(\beta z)| \cdot \left[\left\{ \sum_{n > N} |f_n|^2 \cdot \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + \left\{ \sum_{n > N} |(z - n\pi/\beta) \cdot f'_n|^2 \cdot \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} \right. \\ \left. + \left\{ \sum_{n > N} \left| \frac{(z - n\pi/\beta)^2}{2} \cdot f''_n \right|^2 \cdot \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + \left\{ \sum_{n > N} |f_n|^2 \cdot \sum_{n > N} \left| \frac{1}{2(\beta z - n\pi)} \right|^2 \right\}^{1/2} \right]$$

(10) È. I. Takizawa: Chinese J. Phys. 13, 226 (1975).

(11) J. C. Horng: Chinese J. Phys. 13, 113 (1975).

$$\begin{aligned}
 & + \left\{ \sum_{n < -N} |f_n|^2 \cdot \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + \left\{ \sum_{n < -N} |(z - n\pi/\beta) \cdot f_n'|^2 \cdot \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} \\
 & + \left\{ \sum_{n < -N} \left| \frac{(z - n\pi/\beta)^2}{2} \cdot f_n'' \right|^2 \cdot \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} \\
 & + \left\{ \sum_{n < -N} |f_n|^2 \cdot \sum_{n < -N} \left| \frac{1}{2(\beta z - n\pi)} \right|^2 \right\}^{1/2}, \quad (|\beta z| < N\pi)
 \end{aligned} \tag{44}$$

with

$$|g_n^{(3)}| = 6\beta^3, \quad |g_n^{(4)}| = 0, \quad \text{and} \quad |g_n^{(5)}| = 60\beta^5.$$

Expression (44) leads to:

$$\begin{aligned}
 |E(z)| \leq & |\sin^3(\beta z)| \cdot \left[K_N \cdot \left\{ \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + 2K'_N \cdot \left\{ \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} \right. \\
 & + 2K''_N \cdot \left\{ \sum_{n > N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + K_N \cdot \left\{ \sum_{n > N} \left| \frac{1}{2(\beta z - n\pi)} \right|^2 \right\}^{1/2} \\
 & + L_N \cdot \left\{ \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + 2L'_N \cdot \left\{ \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} \\
 & \left. + 2L''_N \cdot \left\{ \sum_{n < -N} \left| \frac{1}{(\beta z - n\pi)^3} \right|^2 \right\}^{1/2} + L_N \cdot \left\{ \sum_{n < -N} \left| \frac{1}{2(\beta z - n\pi)} \right|^2 \right\}^{1/2} \right],
 \end{aligned} \tag{45}$$

with

$$\left. \begin{aligned}
 K_N &= \sqrt{\sum_{n > N} |f_n|^2}, \quad K'_N = \sqrt{\sum_{n > N} |naf_n'|^2}, \quad K''_N = \sqrt{\sum_{n > N} |(na)^2 \cdot f_n''|^2}, \quad (a = \pi/\beta) \\
 L_N &= \sqrt{\sum_{n < -N} |f_n|^2}, \quad L'_N = \sqrt{\sum_{n < -N} |naf_n'|^2}, \quad L''_N = \sqrt{\sum_{n < -N} |(na)^2 \cdot f_n''|^2}, \quad (a = \pi/\beta)
 \end{aligned} \right\} \tag{46}$$

and

$$\left. \begin{aligned}
 \sqrt{\sum_{n > N} |(z - na) \cdot f_n'|^2} &\leq \sqrt{4 \sum_{n > N} |naf_n'|^2} = 2K'_N, \quad (z < Na) \\
 \sqrt{\sum_{n < -N} |(z - na) \cdot f_n'|^2} &\leq \sqrt{4 \sum_{n < -N} |naf_n'|^2} = 2L'_N, \quad (-Na < z)
 \end{aligned} \right\} \tag{47}$$

also we have:

$$\left. \begin{aligned}
 \sqrt{\sum_{n > N} \left| \frac{(z - na)^2}{2} \cdot f_n'' \right|^2} &\leq \sqrt{4 \sum_{n > N} |(n \cdot a)^2 \cdot f_n''|^2} = 2K''_N, \quad (z < Na) \\
 \sqrt{\sum_{n < -N} \left| \frac{(z - na)^2}{2} \cdot f_n'' \right|^2} &\leq \sqrt{4 \sum_{n < -N} |(na)^2 \cdot f_n''|^2} = 2L''_N, \quad (-Na < z)
 \end{aligned} \right\} \tag{48}$$

Functions $1/(\beta z - u)^6, 1/(\beta z - u)^4$, and $1/(\beta z - u)^2$ in (45) are monotonic decreasing functions of u for $u > \beta z$. We make similar calculations as in (5)~(7), obtaining:

$$\left. \begin{aligned}
 \left\{ \sum_{n > N} \frac{1}{(\beta z - n\pi)^6} \right\}^{1/2} &< \frac{1}{\sqrt{5\pi(N\pi - \beta z)^3}}, \quad \left\{ \sum_{n > N} \frac{1}{(\beta z - n\pi)^4} \right\}^{1/2} < \frac{1}{\sqrt{3\pi(N\pi - \beta z)^3}}, \\
 \left\{ \sum_{n > N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} &< \frac{1}{\sqrt{\pi(N\pi - \beta z)}}, \quad \left\{ \sum_{n < -N} \frac{1}{(\beta z - n\pi)^6} \right\}^{1/2} < \frac{1}{\sqrt{5\pi(\beta z + N\pi)^3}}, \\
 \left\{ \sum_{n < -N} \frac{1}{(\beta z - n\pi)^4} \right\}^{1/2} &< \frac{1}{\sqrt{3\pi(\beta z + N\pi)^3}},
 \end{aligned} \right\} \tag{49}$$

and

$$\left\{ \sum_{n < -N} \frac{1}{(\beta z - n\pi)^2} \right\}^{1/2} < \frac{1}{\sqrt{\pi(\beta z + N\pi)}}. \quad (|\beta z| < N\pi)$$

Putting (49) into (45), we have:

$$|\varepsilon_N''(z)| \leq \frac{|\sin^3(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{K_N + 2K'_N + 2K''_N}{\sqrt{5(N\pi - \beta z)^5}} + \frac{L_N + 2L'_N + 2L''_N}{\sqrt{5(\beta z + N\pi)^5}} \right. \\ \left. + \frac{K_N}{2\sqrt{N\pi - \beta z}} + \frac{L_N}{2\sqrt{\beta z + N\pi}} \right\}. \quad (-N\pi < \beta z < N\pi) \quad (50)$$

$|\varepsilon_N''(z)|$ can be obtained by estimating the quantities K''_N, L''_N . We shall assume that

$$E_k'' \equiv \left\{ \int_{-\infty}^{+\infty} z^{2k} \cdot |z^2 \cdot f''(z)|^2 dz \right\}^{1/2} < +\infty. \quad (51)$$

Here k is a positive integer. Function $f(z)$ is entire, hence $f''(z)$ and $z^{k+2} \cdot f''(z)$ are also entire. Thus we can write (2, 8, 9):

$$z^k \cdot (z^2 \cdot f''(z)) = a^k \cdot \sum_{n=-\infty}^{+\infty} (n)^k \cdot \{(na)^2 \cdot f_n''\} \cdot \frac{\sin(\beta z + \gamma - n\pi)}{\beta z + \gamma - n\pi}. \quad (52)$$

From (12), (51), and (52), we obtain:

$$(E_k'')^2 = \int_{-\infty}^{+\infty} z^{2k} \cdot |z^2 \cdot f''(z)|^2 dz = a^{2k+1} \cdot \sum_{n=-\infty}^{+\infty} (n)^{2k} \cdot |(na)^2 \cdot f_n''|^2. \quad (53)$$

From (53) and (46), we make similar calculations as mentioned in (14)~(17). Thus we have:

$$K_N'' \leq \frac{E_k''}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}; \quad (54)$$

Similarly, we have:

$$L_N'' \leq \frac{E_k''}{a^{k+(1/2)} \cdot (N+1)^k} \cdot \frac{1}{\sqrt{1-4^{-k}}}. \quad (55)$$

Putting (18), (19), (37), (38), (54), and (55) into (50), we have:

$$|\varepsilon_N''(z)| \leq \frac{|\sin^3(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{1}{a^{k+(1/2)} \cdot (N+1)^k \cdot \sqrt{1-4^{-k}}} \right\} \cdot \left\{ \frac{E_k + 2E_k' + 2E_k''}{\sqrt{5(N\pi - \beta z)^5}} \right. \\ \left. + \frac{E_k + 2E_k' + 2E_k''}{\sqrt{5(\beta z + N\pi)^5}} + \frac{E_k}{2\sqrt{N\pi - \beta z}} + \frac{E_k}{2\sqrt{\beta z + N\pi}} \right\}. \quad (-N\pi < \beta z < N\pi) \quad (56)$$

From (56) we obtain:

$$|\varepsilon_N''(z)| \leq \frac{|\sin^3(\beta z)|}{\sqrt{\pi}} \cdot \left\{ \frac{2}{a^{k+(1/2)} \cdot (N+1)^k \cdot \sqrt{1-4^{-k}}} \right\} \\ \cdot \left\{ \frac{E_k + 2E_k' + 2E_k''}{\sqrt{5(N\pi - |\beta z|)^5}} + \frac{E_k}{2\sqrt{N\pi - |\beta z|}} \right\}. \quad (|\beta z| < N\pi) \quad (57)$$

As a special case, we put $z = \pi/(2\beta)$ into (57), and we have:

$$|\varepsilon_N''(\pi/2\beta)| \leq \frac{1}{\pi} \left\{ \frac{\beta}{\pi} \right\}^{k+1/2} \cdot \frac{2}{(N+1)^k \cdot \sqrt{1-4^{-k}}} \cdot \left\{ \frac{E_k + 2E_k' + 2E_k''}{\pi^2 \sqrt{5(N-(1/2))^5}} + \frac{E_k}{2\sqrt{N-(1/2)}} \right\}. \quad (58)$$

As for the numerical examples we shall take constants β, N , and k , to be such values as listed in Table III. The values of the right-hand side of (58) are listed in the last column. These values decrease as N increases.

Table III

β	N	k	The values of the right-hand side of (58)
6	4	1	$7.7 \times 10^{-4} \cdot (E_1 + 2E_1' + 2E_1'') + 1.0 \times 10^{-1} \cdot E_1$
6	6	1	$1.8 \times 10^{-4} \cdot (E_1 + 2E_1' + 2E_1'') + 5.9 \times 10^{-2} \cdot E_1$
6	12	1	$1.5 \times 10^{-5} \cdot (E_1 + 2E_1' + 2E_1'') + 2.2 \times 10^{-2} \cdot E_1$
6	24	1	$1.3 \times 10^{-6} \cdot (E_1 + 2E_1' + 2E_1'') + 8.0 \times 10^{-3} \cdot E_1$

V. CONCLUSION AND ACKNOWLEDGEMENTS

The bounds for truncation error are estimated by using sampling expansions which take into account the sampled zero-th, first, and second order derivatives. The result shows that the value of the right-hand side of (40) is $|\sin(\beta z)| \cdot \{1 + 2(E_k'/E_k)\} / \{\sqrt{3}(N\pi - |\beta z|)\}$ times as large as the value of the same side of (21), when z lies between $-N\pi/\beta$ and $N\pi/\beta$.

In addition, the values of the right-hand side of (22), (41), and (58) decrease when N increases, as are shown in the **Tables I, II, and III**, respectively.

In concluding this paper, the author wishes to express his sincere thanks to Dr. T. S. Lin, President of the Tatung Institute of Technology (Taipei), for his encouragement and financial support, which made it possible for the author to carry out this research in Sapporo.