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A Remark on the Takizawa-Isigaki Generalized Sampling Theorem

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Abstract

Discussions are given on the meaning and the implication of the key assumption in deriving the Takizawa-Isigaki sampling formula.

Résumé

On discute la signification et l'implication de l'assomption-clé dans la dérivation de la formule d'échantillonnage dûe à Takizawa et Isigaki.

IN a recent paper[1], E. Takizawa and H. Isigaki gave the following Theorem. *Let $f(z)$ and $g(z)$ be entire analytic functions* in *the complex plane C. Assume that*

(A) $\lim_{z\to\infty} f(z)/g(z)=0.$

Then

(F) $$f(z)=\sum_n \sum_{s=0}^{m_n} \sum_{j=0}^{s} K_{njs}(z-z_n)^{s-m_n-1}g(z).$$

Here z_n are zeros of $g(z)$ (of order m_n+1) and

$$K_{njs}=f_n^{(j)}H_n^{(s-j)}/j!\,(s-j)\,!$$

with $f_n^{(j)}, H_n^{(s-j)}$ constants depending on f and g.

Their proof relies on the evaluation of a contour integral

(I) $$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)g(\zeta)}d\zeta$$

on a certain Jordan curve γ in the complex plane C. The assumption (A) is employed to evaluate the integral (I) when the diameter of γ grows to infinity. However, when taking in mind the applications, and so validity, of formula (F), we must pay attention to the implication of the assumption (A).

We claim the assumption (A) is equivalent to that $f(z)/g(z)$ is a rational function, that is,

$$f(z)/g(z)=P(z)/Q(z),$$

(1) E. I. Takizawa and H. Isigaki: Memo. Fac Engng. Hokkaido Univ. 13, 281-295, (1974).

$P(z)$, $Q(z)$ being polynomials in z, the degree of P smaller than that of Q. In fact, (A) means that given any $\varepsilon>0$ we can find $R_\varepsilon>0$ such that

$$|f(z)|\leqq\varepsilon|g(z)| \quad \text{for} \quad |z|>R_\varepsilon.$$

Therefore, if z_0 is a zero of $g(z)$ of order m with $|z_0|>R_\varepsilon$, then z_0 must be a zero of $f(z)$ of order at least m. Thus, the function $f(z)/g(z)$ has no poles in $|z|>R_\varepsilon$. Furthermore, after a change of variable $w=1/z$, the function $F(w)=f(1/w)/g(1/w)$, which is holomorphic in $0<|w|<1/R_\varepsilon$, is bounded near $w=0$. Namely, $w=0$ is a removable singularity for $F(w)$. If we set $F(0)=0$, then $F(w)$ is analytic in $|w|<1/R_\varepsilon$.
Hence, the function $f(z)/g(z)$ has only a finite number of poles, and is holomorphic at infinity. This means that $f(z)/g(z)$ is written in the form $f_1(z)/Q(z)$ with a polynomial $Q(z)$ and an entire function $f_1(z)$. By the assumptions (A),

$$|f_1(z)|\leqq\varepsilon|Q(z)| \quad \text{for} \quad |z|>R_\varepsilon.$$

So, by Liouville's theorem, $f_1(z)$ is a polynomial $P(z)$. Using again the assumption (A), we see

the degree of P < the degree of Q.

Similar considerations allow us to weaken the assumption (A) in the following form:

(A′) $\qquad \lim_{z\to\infty} f(z)/z^\nu g(z)=0,$

ν being a fixed positive integer. The function $f(z)/g(z)$ is rational even under the assumption (A′). This time, however, $f(z)/g(z)$ has a pole at infinity of order at most $\nu-1$.

For $\zeta\in C$ fixed, the function $f(z)/(z-\zeta)g(z)$ is rational under the assumptions **(A)** or **(A′)**. Thus, considering on the Riemann sphere $C\cup\{\infty\}$ on which the sum of the residues of any rational function vanishes[2], we can obtain a slight generalization of the formula (F) (by adding the residue at infinity when we take the assumption (A′)). Of course, in any case, the summation over n in formula **(F)** is a finite sum. We leave the reader to derive exact formulas.

(2) H. Cartan: *Théorie Elémentaire des Fonctions Analytiques d'une ou Plusieurs* Variables Complexes (Hermann, Paris, 1961).